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Valuations: Bi, Tri, and Tetra

Abstract. This paper considers some issues to do with *valuational* presentations of consequence relations, and the Galois connections between spaces of valuations and spaces of consequence relations. Some of what we present is known, and some even well-known; but much is new. The aim is a systematic overview of a range of results applicable to nonreflexive and nontransitive logics, as well as more familiar logics. We conclude by considering some connectives suggested by this approach.

Keywords: Valuational semantics, Nontransitive logic, Nonreflexive logic.

This paper considers some issues to do with *valuational* presentations of consequence relations, and the Galois connections between spaces of valuations and spaces of consequence relations. Our core inspirations are [33, 49, 50], but we draw on a range of other work throughout. Some of what we present is known, and some even well-known; but much is new.

Our aim is a systematic overview, so we do not spend time pursuing any particular applications; but we have been brought to this work through its applications. Thinking about semantic paradoxes has pushed one of us into exploring nonreflexive logics [25], and the other into exploring nontransitive logics [44, 45]. So we are interested in adapting techniques developed for exploring more familiar logics into these less familiar domains. But we will not comment further in this paper on these background motivations. Our aims here are simply to show that some familiar techniques indeed do generalize quite simply; applying (and interpreting) these techniques can come elsewhere.

Here's the plan. In Section 1, we introduce Galois connections between sets of valuations and sets of arguments, and their associated closure operations, before setting out the particular Galois connections that stand at the heart of the paper. In Section 2, we charactize the sets of arguments and sets of valuations that are closed with regard to (the closure operations induced by) these Galois connections, and briefly describe the lattice structures that these closed sets inhabit. Finally, in Section 3, we consider the effects of introducing vocabulary into our language that witnesses the structure of the valuations in play.

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1. Arguments and Valuations

At the center of the valuational approach to consequence there are three components: some set $\mathfrak{V}$ of *valuations*, some set $\mathfrak{A}$ of *arguments*, and a binary *counterexample* relation $\divideontimes$ from $\mathfrak{V}$ to $\mathfrak{A}$.

Sometimes, both $\mathfrak{V}$ and $\mathfrak{A}$ are determined by some *language* $\mathcal{L}$, with $\mathfrak{V}$ set to be $\mathcal{T}^{\mathcal{L}}$ for some set $\mathcal{T}$ of values, and $\mathfrak{A}$ set to be $\wp(\mathcal{L}) \times \mathcal{L}$, or $\wp(\mathcal{L}) \times \wp(\mathcal{L})$, or something of the kind. Indeed, this way of arriving at $\mathfrak{V}$ and $\mathfrak{A}$ will involve us for the bulk of the paper, with a particular eye on four possible choices for $\mathcal{T}$. But just for now, let's put off thinking about all this extra structure, and see what comes simply from thinking in terms of counterexamples, without any attention at all to what kinds of things arguments or valuations might be.

On a standard way of thinking, which we will work with throughout, an argument is *valid* iff it *has no counterexamples*. This means that each set of valuations $V \subseteq \mathfrak{V}$, together with some counterexample relation between valuations and arguments, determines a set of arguments $\mathcal{A}(V) \subseteq \mathfrak{A}$: the set of arguments that have no counterexamples in V. By the same lights, although this is less frequently emphasized, each set of arguments $A \subseteq \mathfrak{A}$, together with a counterexample relation, determines a set of valuations $\mathcal{V}(A) \subseteq \mathfrak{V}$: the set of valuations that are not a counterexample to any argument in A.

This instantiates a general and well-explored structure: that of a Galois connection. Whenever we have two sets S and T with a binary relation R from S to T, this relation induces two functions $f : \wp(S) \to \wp(T)$ and $g : \wp(T) \to \wp(S)$ as follows:

$$f(X) = \{t \in T : \forall s \in X, sRt\}$$
$$g(Y) = \{s \in S : \forall t \in Y, sRt\}$$

such that for all $X \subseteq S$ and $Y \subseteq T$, we have $X \subseteq g(Y)$ iff $Y \subseteq f(X)$. This last condition is the Galois condition, and any f, g satisfying this condition are called a *Galois connection* between $\wp(S)$ and $\wp(T)$.[1] Galois connections

[1] An example due to [34, p. 4]: think of the relation "has visited" between people and cities. Then for any set X of people, $f(X)$ is the set of cities that all of them have visited in common; and for any set Y of cities, $g(Y)$ is the set of people that have visited all those cities. So whenever $X \subseteq g(Y)$, the people in X are among the people that have visited every city in Y. Thus, $Y \subseteq f(X)$; the cities in Y are among the cities that everyone in X has visited. The converse follows similarly.

provide a great deal of useful structure, and we will cite and exploit a range of their properties in what follows.[2]

In the case of valuations and arguments, the key relation that fits this form is the *complement* of the counterexample relation: the relation that obtains between a valuation and an argument when the valuation is not a counterexample to that argument.[3] So, for any sets V of valuations and A of arguments, we have $V \subseteq \mathcal{V}(A)$ iff $A \subseteq \mathcal{A}(V)$: each is true iff nothing in V is a counterexample to anything in A. From this, all of the following follows:

Theorem 1. *(Galois facts) For any $V, V' \subseteq \mathfrak{V}$ and $A, A' \subseteq \mathfrak{A}$,*

(i) *if $V \subseteq V'$, then $\mathcal{A}(V) \supseteq \mathcal{A}(V')$,*

(ii) *if $A \subseteq A'$, then $\mathcal{V}(A) \supseteq \mathcal{V}(A')$,*

(iii) *$\mathcal{V} \circ \mathcal{A}$ (henceforth, $\mathcal{VA}$) is a closure operation on $\langle \wp(\mathfrak{V}), \subseteq \rangle$,*[4]

(iv) *$\mathcal{A} \circ \mathcal{V}$ (henceforth, $\mathcal{AV}$) is a closure operation on $\langle \wp(\mathfrak{A}), \subseteq \rangle$,*

(v) *$\mathcal{V}(A)$ is closed wrt $\mathcal{VA}$,*

(vi) *$\mathcal{A}(V)$ is closed wrt $\mathcal{AV}$, and*

(vii) *$\mathcal{A}$ and $\mathcal{V}$ form an (order-inverting) isomorphism between the closed elements of $\wp(\mathfrak{V})$ and the closed elements of $\wp(\mathfrak{A})$.*

Proof. See for example [7,8,16,18,20,40].[5] ■

Within $\mathfrak{V}$ and $\mathfrak{A}$, then, some subsets are distinguished by the Galois connection. Some sets $V \subseteq \mathfrak{V}$ are *closed*: such that $V = \mathcal{VA}(V)$. Since $\mathcal{VA}$ is a closure operation, always $V \subseteq \mathcal{VA}(V)$; what is special about closed

[2] The Galois condition makes sense if stated in terms of any partial order, not just $\subseteq$. Also, Galois connections come in *antitone* and *monotone* versions; we are here using the (original) antitone version. (The condition for the monotone version is that $X \leq g(Y)$ iff $f(X) \leq Y$.) These are essentially the same thing, however: a monotone Galois connection between S and T is exactly an antitone Galois connection between S and the order-dual of T. For helpful discussion on this difference, see [18].

[3] We do not often concern ourselves with the questions: given this set of valuations, which arguments are counterexampled *by each* valuation in the set? or given this set of arguments, which valuations manage to be counterexamples to *all* of them? These are the questions we would focus on if we applied this way of generating a Galois connection to the counterexample relation itself, rather than its complement.

[4] A *closure* operation on a partially-ordered set $\langle S, \leq \rangle$ is an operation C such that for every $X, Y \in S$: 1) $X \leq C(X)$; 2) if $X \leq Y$, then $C(X) \leq C(Y)$; and 3) $C(C(X)) \leq C(X)$. (Equivalently, such that for every $X, Y \in S$: $X \leq C(Y)$ iff $C(X) \leq C(Y)$.) An $X \in S$ is *closed* wrt C iff $X = C(X)$.

[5] But note that [16,20] use the monotone understanding of Galois connection rather than the antitone one; recall Footnote 2.

V is the inclusion in the other direction. A closed V contains *every* valuation compatible with everything in $\mathcal{A}(V)$: adding any valuation to a closed V that it does not already contain would result in a V' such that $\mathcal{A}(V) \neq \mathcal{A}(V')$. By contrast, if V is not closed there is some $v \notin V$ such that $\mathcal{A}(V \cup \{v\}) = \mathcal{A}(V)$, some way V could be more inclusive without having any counterexamples to any new arguments.

And all this plays out on the other side as well: some sets $A \subseteq \mathfrak{A}$ are *closed*: such that $A = \mathcal{AV}(A)$. Since $\mathcal{AV}$ is a closure operation, always $A \subseteq \mathcal{AV}(A)$; what is special about closed A is the inclusion in the other direction. A closed A contains *every* argument that lacks a counterexample in $\mathcal{V}(A)$: adding any argument to a closed A that is does not already contain would result in an A' such that $\mathcal{V}(A) \neq \mathcal{V}(A')$. By contrast, if A is not closed there is some $a \notin A$ such that $\mathcal{V}(A \cup \{a\}) = \mathcal{V}(A)$, some way A could be more inclusive without ruling out any additional valuations.

1.1. One Layer Down

All that is perfectly general. But when we apply these ideas to actual logical situations, there is often more structure in play to take advantage of. By making arguments more argumenty and valuations more valuationy, we get more handles to grab on to. Importantly, this extra structure productively interacts with the general background we have given above. In the remainder of the paper, we follow the approach we briefly flagged earlier, of taking $\mathfrak{V}$ and $\mathfrak{A}$ to each be determined from some underlying language $\mathcal{L}$. For now, we make no assumptions about any structure exhibited by $\mathcal{L}$, considering it merely as a set, and referring to its members as *formulas*. We make no assumptions in general about the cardinality of $\mathcal{L}$ (although see Fact 4 for one place its cardinality can matter). We use ϕ, ψ and the like for formulas, and Γ, Δ, Σ and the like for sets of formulas.

1.1.1. Eight Galois Connections Given this language, we consider two possible background sets $\mathfrak{A}$ of arguments, and four possible background sets $\mathfrak{V}$ of valuations. (There is a sense in which we consider only one possible counterexample relation, although it needs to be restated depending on which set of arguments is in play.) Either of the sets of arguments can be Galois connected to any of the sets of valuations, so this means we will be exploring eight Galois connections.

On the argument side, we consider two different candidates for $\mathfrak{A}$. The first, the set of Set-Set arguments, is $\mathfrak{A}_{\text{SS}} = \wp(\mathcal{L}) \times \wp(\mathcal{L})$. These are arguments with a *set* of premises and a *set* of conclusions. We write the Set-Set argument $\langle \Gamma, \Delta \rangle$ as $[\Gamma \succ \Delta]$, abbreviating in all kinds of usual sequent-calculus ways. The second, the set of SET-FMLA arguments, is $\mathfrak{A}_{SF} = \wp(\mathcal{L}) \times \mathcal{L}$. These are arguments with a set of premises and a *single* formulas as a conclusion. We write the SET-FMLA argument $\langle \Gamma, \phi \rangle$ as $[\Gamma \rhd \phi]$, again abbreviating in all kinds of usual sequent-calculus ways. Note that there is no restriction to *finite* arguments here; where sets of formulas are involved, they may be of any cardinality that $\mathcal{L}$ provides.

We refer to SET-FMLA and SET-SET as *frameworks* (following the usage of that term in [34, pp. 103–112]), and do not mix them; when we speak of a 'set of arguments' in the sequel, each such set should be taken to be a subset either of $\mathfrak{A}_{SS}$ or $\mathfrak{A}_{SF}$; and when we make reference to $\mathfrak{A}$, this too should be understood as one of $\mathfrak{A}_{SS}$ or $\mathfrak{A}_{SF}$. (Sometimes we will mean one or the other, and other times we are being deliberately neutral; we trust context to clarify.)

There is an important (if kind of obvious) partial order $\sqsubseteq$ on arguments.

DEFINITION 1. For SET-SET arguments, $[\Gamma \succ \Delta] \sqsubseteq [\Gamma' \succ \Delta']$ iff $\Gamma \subseteq \Gamma'$ and $\Delta \subseteq \Delta'$; and for SET-FMLA arguments, $[\Gamma \rhd \phi] \sqsubseteq [\Gamma' \rhd \psi]$ iff $\Gamma \subseteq \Gamma'$ and $\phi = \psi$.

On the valuational side, we begin with a familiar structure—but we encourage you, at least for now, not to treat it as too familiar, since we're going to make use of it in what may be a less familiar way. The structure in question is the set $\{\top, \bot, \mathbb{I}, *\}$, equipped with two partial orders $\sqsubseteq$ and $\preccurlyeq$, as depicted in Figure 1. We call the members of this set *values*.[6] We refer to the order $\sqsubseteq$ as the *information order*, and to the order $\preccurlyeq$ as the *truth order*.

As this paper's title suggests, we are not primarily concerned with values but rather with valuations—functions from $\mathcal{L}$ to values—and with various kinds of valuations. We consider four candidate sets of valuations to fill in the role of $\mathfrak{V}$. The first, the set of *tetravaluations*, is $\mathfrak{V}_4 = \{\top, \bot, \mathbb{I}, *\}^{\mathcal{L}}$. The second, the set of *reflexive trivaluations*, is $\mathfrak{V}_3^r = \{\top, \bot, *\}^{\mathcal{L}}$. The third, the set of *transitive trivaluations*, is $\mathfrak{V}_3^t = \{\top, \bot, \mathbb{I}\}^{\mathcal{L}}$. And the final candidate, the set of *bivaluations*, is $\mathfrak{V}_2 = \{\top, \bot\}^{\mathcal{L}}$. Clearly $\mathfrak{V}_3^r, \mathfrak{V}_3^t \subseteq \mathfrak{V}_4$, and $\mathfrak{V}_2 = \mathfrak{V}_3^r \cap \mathfrak{V}_3^t$.[7] In a context where we are discussing only a particular one of these sets of valuations, 'values' should be understood as restricted

[6] Sometimes these values are denoted '**T**', '**F**', '**B**', and '**N**', as in [51]; and sometimes they have other names, as in [46]. Here, we employ the notation of [10], with the addition of '$\mathbb{I}$' for the value $\mathbb{I}$.

[7] It is not true, however, that $\mathfrak{V}_4 = \mathfrak{V}_3^r \cup \mathfrak{V}_3^t$: there are tetravaluations that use all four values, and are thus neither kind of trivaluation.

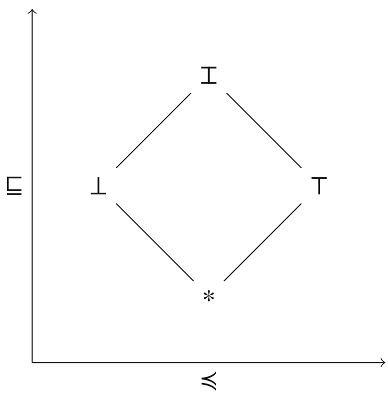

Figure 1. Two orders on values

to the corresponding values. When we make reference to $\mathfrak{V}$, this should be understood as one of these, or as neutral among them, according to context.

The partial orders $\sqsubseteq$ and $\preccurlyeq$ on values can be lifted pointwise to partial orders on valuations: so for valuations u, v we have $u \sqsubseteq v$ iff for all $\phi \in \mathcal{L}$, $u(\phi) \sqsubseteq v(\phi)$, and similarly for $\preccurlyeq$. It is customary to note that both $\sqsubseteq$ and $\preccurlyeq$ are lattice orders on $\{\top, \bot, \top\!\!\!\!\bot, *\}$, and so our values so ordered form a (very small) *bilattice*.[8] As a result, $\mathfrak{V}_4$ also forms a (much larger, depending on $|\mathcal{L}|$) bilattice under the lifted orders. We will make some use of this later in the paper, but for now we simply note that this does not apply to our more restricted sets of valuations: in particular, the lifted $\sqsubseteq$ fails to be a lattice order on all of $\mathfrak{V}_3^r$, $\mathfrak{V}_3^t$, and $\mathfrak{V}_2$, and so the theory of bilattices is of more limited use in exploring these restricted sets of valuations.

With this much in place, we can state the counterexample relation we are interested in. This is the only candidate we will consider, and will form the core of the investigations to follow:

Definition 2. A valuation v is a *counterexample* to an argument $a = [\Gamma \succ \Delta]$ (in the Set-Fmla framework $a = [\Gamma \rhd \phi]$)—written $v \mathbin{※} a$—iff $v[\Gamma] \subseteq \{\top, \top\!\!\!\!\bot\}$ and $v[\Delta] \subseteq \{\bot, \top\!\!\!\!\bot\}$ (in the Set-Fmla framework $v(\phi) \in \{\bot, \top\!\!\!\!\bot\}$).

For bivaluations, this is completely usual: a counterexample assigns value $\top$ to all premises and value $\bot$ to all conclusions, with these the only two values in play. However, when we move beyond bivaluations, there are other possibilities: $\top\!\!\!\!\bot$ plays the counterexampling roles of both $\top$ and $\bot$ (hence the

[8]For more on lattices, see [16]. For more on bilattices, see Section 3, where we give additional references.

symbol $\top\!\!\!\!\bot$), and $*$ provides a value that plays no role in counterexampling at all.[9] In the terminology of, for example [53, p. 172], $\top$ and $\top\!\!\!\!\bot$ are *designated*, and $\bot$ and $\top\!\!\!\!\bot$ are *antidesignated*.

This is a natural notion of a counterexample to consider.[10] To see this, consider a different abstract approach to semantics, following [50, pp. 14ff, 28ff]. Instead of valuations, this approach uses pairs $\langle \boldsymbol{T}, \boldsymbol{U} \rangle$ with $\boldsymbol{T}, \boldsymbol{U} \subseteq \mathcal{L}$; we can say that $\langle \boldsymbol{T}, \boldsymbol{U} \rangle \divideontimes [\Gamma \succ \Delta]$ (or $\langle \boldsymbol{T}, \boldsymbol{U} \rangle \divideontimes [\Gamma \rhd \boldsymbol{\phi}]$) iff $\Gamma \subseteq \boldsymbol{T}$ and $\Delta \subseteq \boldsymbol{U}$ (alternatively, $\Gamma \subseteq \boldsymbol{T}$ and $\boldsymbol{\phi} \in \boldsymbol{U}$). Shoesmith and Smiley [50] considers only those pairs $\langle \boldsymbol{T}, \boldsymbol{U} \rangle$ with $\boldsymbol{T} \cap \boldsymbol{U} = \emptyset$ and $\boldsymbol{T} \cup \boldsymbol{U} = \mathcal{L}$, but this restriction can be removed. If we do consider arbitrary pairs $\langle \boldsymbol{T}, \boldsymbol{U} \rangle$, then there are four different statuses which a formula $\boldsymbol{A}$ could have wrt any such pair: it could be in both $\boldsymbol{T}$ and $\boldsymbol{U}$ (alias $v(\boldsymbol{A}) = \top\!\!\!\!\bot$), in neither $\boldsymbol{T}$ nor $\boldsymbol{U}$ (alias $v(\boldsymbol{A}) = *$), only in $\boldsymbol{T}$ (alias $v(\boldsymbol{A}) = \top$), or only in $\boldsymbol{U}$ (alias $v(\boldsymbol{A}) = \bot$). On this approach, we can see $\mathfrak{V}_4$ as the set of all pairs; $\mathfrak{V}_3^r$ as the set of pairs $\langle \boldsymbol{T}, \boldsymbol{U} \rangle$ with $\boldsymbol{T} \cap \boldsymbol{U} = \emptyset$; $\mathfrak{V}_3^t$ as the set of pairs $\langle \boldsymbol{T}, \boldsymbol{U} \rangle$ with $\boldsymbol{T} \cup \boldsymbol{U} = \mathcal{L}$; and finally $\mathfrak{V}_2$ as the set of pairs $\langle \boldsymbol{T}, \boldsymbol{U} \rangle$ with $\boldsymbol{T} \cap \boldsymbol{U} = \emptyset$ and $\boldsymbol{T} \cup \boldsymbol{U} = \mathcal{L}$. (Shoesmith and Smiley [50], then, considers exactly $\mathfrak{V}_2$, on this understanding.) Everything we do with valuations, then, can just as easily be done with such pairs.

We now have enough on the table to generate the Galois connections we are interested in, fitting our opening presentation. Given any set $\boldsymbol{V}$ of valuations (of any sort), we let $\mathcal{A}_{\text{SS}}(\boldsymbol{V})$ be the set of Set-Set arguments with no counterexample in $\boldsymbol{V}$, and $\mathcal{A}_{\text{SF}}(\boldsymbol{V})$ be the set of Set-Fmla arguments with no counterexample in $\boldsymbol{V}$. When we wish to remain framework-neutral, we will speak of $\mathcal{A}(\boldsymbol{V})$. Similarly, given any set $\boldsymbol{A}$ of arguments (in either framework), we let $\mathcal{V}_4(\boldsymbol{A})$ be the set of tetravaluations that do not counterexample any argument in $\boldsymbol{A}$, with $\mathcal{V}_3^r(\boldsymbol{A})$, $\mathcal{V}_3^t(\boldsymbol{A})$, and $\mathcal{V}_2(\boldsymbol{A})$ similar for reflexive trivaluations, transitive trivaluations, and bivaluations, respectively. Note that, no matter what $\boldsymbol{A}$ is, $\mathcal{V}_3^r(\boldsymbol{A}) = \mathcal{V}_4(\boldsymbol{A}) \cap \mathfrak{V}_3^r$; $\mathcal{V}_3^t(\boldsymbol{A}) = \mathcal{V}_4(\boldsymbol{A}) \cap \mathfrak{V}_3^t$; and $\mathcal{V}_2(\boldsymbol{A}) = \mathcal{V}_4(\boldsymbol{A}) \cap \mathfrak{V}_2$. This leaves us with eight Galois connections, one for each choice of framework and kind of valuation.

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[9]The role of $*$ is perhaps most familiar from 'weak Schütte valuations', a special type of reflexive trivaluation, discussed for example in [28, Ch. 3; 31]. The latter source also considers 'strong Schütte valuations', which emerge as a special type of transitive trivaluation. (Hösli and Jäger [31] uses the single symbol '**u**' for both $*$ and $\top\!\!\!\!\bot$, giving two different understandings of what it is for a valuation to be a counterexample to an argument.)

[10]A related notion of consequence is explored in the Fmla-Fmla-fragment in [52, Proposition 1]. The approach considered there involves a richer language than that which is operative in most of the present paper, essentially amounting to the language of Section 3 containing only the truth-connectives $\curlywedge$, $\curlyvee$, and $\sim$ (i.e. what in Section 3.4 is $\mathcal{L}_{\mathcal{S}}$ for $\mathcal{S} = \{\curlywedge, \curlyvee, \sim\}$).

This counterexample relation has particularly pleasant interactions with the information order on valuations and the argument order. These interactions will come in for heavy use in what follows:

FACT 1. *For any arguments* a, b *and valuations* v, w*: if* $v \divideontimes a$ *and* $v \sqsubseteq w$ *and* $b \sqsubseteq a$*, then* $w \divideontimes b$.

PROOF. Unpacking definitions. ∎

1.1.2. Exact Counterexamples Among the valuations that serve as counterexamples to an argument, some are what we will call *exact*.

DEFINITION 3. A valuation v is an *exact counterexample* to an argument a—written $v \bowtie a$—iff for all arguments $b \in \mathfrak{A}$, $v \divideontimes b$ iff $b \sqsubseteq a$.

DEFINITION 4. A valuation v is *targeted* iff there is exactly one formula ϕ such that $v(\phi) \in \{\bot, \mathbb{I}\}$; in such a case, v's *target* is ϕ.

FACT 2. $\bowtie$ *is a bijection between* $\mathfrak{V}_4$ *and* $\mathfrak{A}_{\text{SS}}$*, and between the targeted tetravaluations and* $\mathfrak{A}_{\text{SF}}$.

PROOF. We cannot have $v \bowtie a$ and $v \bowtie b$ for $a \neq b$: in this case either $a \not\sqsubseteq b$ or $b \not\sqsubseteq a$. Wlog, suppose $a \not\sqsubseteq b$; then since $v \bowtie b$ we can conclude that v is not even a counterexample to a, let alone an exact one. Similarly, we cannot have $u \bowtie a$ and $v \bowtie a$ for $u \neq v$, since if $u \neq v$ there must be some argument counterexampled by one but not the other. So exact counterexampling is one-one in both directions. It remains to show that it is total.

$\mathfrak{A}_{\text{SS}}$: Given an argument $a = [\Gamma \succ \Delta]$, let $v(\phi) = \mathbb{I}$ iff $\phi \in \Gamma \cap \Delta$; otherwise $\top$ iff $\phi \in \Gamma$ and $\bot$ iff $\phi \in \Delta$; otherwise $*$. Given a valuation v, let $a = [\{\phi \mid v(\phi) \in \{\top, \mathbb{I}\}\} \succ \{\phi \mid v(\phi) \in \{\bot, \mathbb{I}\}\}]$. Either way, cashing out definitions reveals that $v \bowtie a$.

$\mathfrak{A}_{\text{SF}}$: Given an argument $a = [\Gamma \rhd \psi]$, let $v(\phi) = \mathbb{I}$ iff $\phi = \psi \in \Gamma$; otherwise $\top$ iff $\phi \in \Gamma$ and $\bot$ iff $\phi = \psi$; otherwise $*$. Note that v so defined is always targeted. Given a targeted valuation v, let $a = [\{\phi \mid v(\phi) \in \{\top, \mathbb{I}\}\} \rhd \psi]$, where ψ is v's target. Either way, cashing out definitions reveals that $v \bowtie a$. ∎

In light of Fact 2, we refer a few times in what follows to 'the exact counterexample of a' or 'the argument exactly counterexampled by v', intending in each case to appeal to this bijection.

2. Closed Sets

The Galois connection between sets of arguments and sets of valuations gives a valuable tool for exploring both spaces and the relations between them. It is particularly suited for exploring the *closed* sets of arguments and valuations, since these are exactly those sets in the range of the functions constituting the connection, and these functions give an (order-inverting) isomorphism between the closed sets of arguments and the closed sets of valuations.

However, it is important to remember that a set of arguments or valuations is closed only with respect to a particular Galois connection. For example, there are sets of transitive trivaluations that are $\mathcal{V}_3^t\mathcal{A}_{\text{SS}}$-closed but not $\mathcal{V}_4\mathcal{A}_{\text{SS}}$-closed. (Indeed, no nonempty $\mathcal{V}_3^t\mathcal{A}_{\text{SS}}$-closed set of valuations is $\mathcal{V}_4\mathcal{A}_{\text{SS}}$-closed, as we will implicitly see in footnote 13.)

Of course, there is an easy way to characterize those sets that are closed wrt a particular Galois connection: just specify the Galois connection in question and then say 'closed'. But these Galois connections are of use, in large part, because there are alternate characterizations of the closed sets that are tractable in other ways. Here, we explore some of these alternate characterizations.

2.1. Closed Sets of Arguments

Definition 5. A set A of arguments is:

- *reflexive* iff for each $\phi \in \mathcal{L}$, (Set-Set:) $[\phi \succ \phi] \in A$, or (Set-Fmla:) $[\phi \rhd \phi] \in A$;
- *monotonic* iff whenever $a \in A$ and $a \sqsubseteq b$, then $b \in A$;
- (Set-Set:) *completely transitive* iff for all $\Sigma \subseteq \mathcal{L}$, if for all $\Sigma_1 \cup \Sigma_2 = \Sigma$, $[\Sigma_1, \Gamma \succ \Delta, \Sigma_2] \in A$, then $[\Gamma \succ \Delta] \in A$; and
- (Set-Fmla:) *completely transitive* iff for all $\Sigma \subseteq \mathcal{L}$, if $[\Gamma \rhd \sigma] \in A$ for each $\sigma \in \Sigma$ and $[\Sigma, \Gamma \rhd \phi] \in A$, then $[\Gamma \rhd \phi] \in A$.

Reflexivity and complete transitivity are framework-dependent; they come in different versions, each suited to be applied within a particular framework.[11] When we call a set of arguments 'completely transitive', then,

[11] In the case of reflexivity, it causes no real trouble to ignore this. 'Complete' transitivity is so-called in order to distinguish it (or at least gesture at a distinction) from various other transitivity-like properties a set of arguments might have. For partial exploration

we always mean the version appropriate to the framework the set of arguments works within.

These structural properties are the keys to closed sets of arguments.

Theorem 2. *A set of arguments is:*

- $\mathcal{AV}_4$*-closed iff it is monotonic,*
- $\mathcal{AV}_3^r$*-closed iff it is monotonic and reflexive,*
- $\mathcal{AV}_3^t$*-closed iff it is monotonic and completely transitive, and*
- $\mathcal{AV}_2$*-closed iff it is monotonic, reflexive, and completely transitive.*

Proof. Recall that A is closed iff $A = \mathcal{AV}(A)$. In each case, we proceed as follows: by showing first that if $A = \mathcal{A}(V)$ for *any* set V of the appropriate kind, then it has the needed structural properties, and showing second that if A has the needed structural properties, then $\mathcal{AV}(A) \subseteq A$. (For any set of arguments A, we already have $A \subseteq \mathcal{AV}(A)$ by Theorem 1.)

$\mathcal{V}_4$

LTR Suppose $A = \mathcal{A}(V)$ for any $V \subseteq \mathfrak{V}_4$, to show A is monotonic. Suppose $b \notin A$ and $a \sqsubseteq b$. Since $b \notin \mathcal{A}(V)$, there must be some $v \in V$ with $v \divideontimes b$. But then $v \divideontimes a$ as well, and so $a \notin A$ either.

RTL Set-Set: Suppose A is monotonic, to show $\mathcal{AV}_4(A) \subseteq A$. Suppose $[\Gamma \succ \Delta] \notin A$. Then it must be that there is no $b \in A$ with $b \sqsubseteq [\Gamma \succ \Delta]$. Let v be the exact counterexample of $[\Gamma \succ \Delta]$; then for any argument a, $v \divideontimes a$ iff $a \sqsubseteq [\Gamma \succ \Delta]$. And we know there is no such $a \in A$, so $v \in \mathcal{V}_4(A)$. But $v \divideontimes [\Gamma \succ \Delta]$, so $[\Gamma \succ \Delta] \notin \mathcal{AV}_4(A)$.
Set-Fmla: similar.

$\mathcal{V}_3^r$

LTR Suppose $A = \mathcal{A}(V)$ for any $V \subseteq \mathfrak{V}_3^r$, to show A is monotonic and reflexive. Since $V \subseteq \mathfrak{V}_4$, it follows from the above that A is monotonic. Since $V \subseteq \mathfrak{V}_3^r$, for any $\phi \in \mathcal{L}$ and $v \in V$, $v(\phi) \in \{\top, \bot, *\}$.

Footnote 11 continued
of this space of properties (in both Set-Fmla and Set-Set frameworks, but assuming monotonicity throughout), see [47].

The Set-Fmla form is exactly the property called 'cut for sets' for the Set-Fmla framework in [50, p. 15]; the Set-Set form, however, is *not* the property called 'cut for sets' for the Set-Set framework on p. 29 there. Our version of the property is *weaker*. However, it is equivalent to Shoesmith & Smiley's property in the presence *either* of monotonicity or of the property sometimes called 'overlap', which holds of a set of arguments iff it contains every argument of the form $[\Gamma, \phi \succ \phi, \Delta]$. (In the present setting, unlike some others, monotonicity and overlap are independent of each other.)

But in none of these three cases do we have $v \divideontimes [\phi \succ \phi]$, so A is reflexive too.

RTL Suppose A is monotonic and reflexive. Then by the $\mathcal{V}_4$ case we have $\mathcal{A}\mathcal{V}_4(A) \subseteq A$. We claim $\mathcal{V}_4(A) = \mathcal{V}_3^r(A)$, from which we get $\mathcal{A}\mathcal{V}_3^r(A) \subseteq A$ immediately. If our claim is wrong, there is some $v \in \mathcal{V}_4(A)$ and $\phi \in \mathcal{L}$ with $v(\phi) = \mathbb{I}$; but then $v \divideontimes [\phi \succ \phi]$ and so $[\phi \succ \phi] \notin \mathcal{A}\mathcal{V}_4(A)$. It follows that $[\phi \succ \phi] \notin A$, and so A is not reflexive, which contradicts our supposition.

Set-Fmla: similar.

$\mathcal{V}_3^t$

LTR Suppose $A = \mathcal{A}(V)$ for some $V \subseteq \mathfrak{V}_3^t$, to show A is monotonic and completely transitive. Since $V \subseteq \mathfrak{V}_4$, it follows from the above that A is monotonic.

Set-Set: Suppose there is some $v \in V$ with $v \divideontimes [\Gamma \succ \Delta]$. We must then show that for any $\Sigma \subseteq \mathcal{L}$, there are some $\Sigma_1 \cup \Sigma_2 = \Sigma$ with $[\Sigma_1, \Gamma \succ \Delta, \Sigma_2] \notin \mathcal{A}(V)$. So take any $\Sigma \subseteq \mathcal{L}$, and let $\Sigma_1 = \{\phi \in \Sigma | s.v(\phi) \in \{\top, \mathbb{I}\}\}$ and $\Sigma_2 = \{\phi \in \Sigma | s.v(\phi) \in \{\bot, \mathbb{I}\}\}$. Since $V \subseteq \mathfrak{V}_3^t$, $\Sigma_1 \cup \Sigma_2 = \Sigma$; but $v \divideontimes [\Sigma_1, \Gamma \succ \Delta, \Sigma_2]$, so $[\Sigma_1, \Gamma \succ \Delta, \Sigma_2] \notin \mathcal{A}(V)$.

Set-Fmla: Suppose there is some $v \in V$ with $v \divideontimes [\Gamma \rhd \phi]$. We must then show that for any $\Sigma \subseteq \mathcal{L}$, either $[\Sigma, \Gamma \rhd \phi] \notin \mathcal{A}_{\text{SF}}(V)$ or else for some $\sigma \in \Sigma$, $[\Gamma \rhd \sigma] \notin \mathcal{A}_{\text{SF}}(V)$. So take any $\Sigma \subseteq \mathcal{L}$; since $V \subseteq \mathfrak{V}_3^t$ either $v[\Sigma] \subseteq \{\top, \mathbb{I}\}$ or there is some $\sigma \in \Sigma$ with $v(\sigma) \in \{\bot, \mathbb{I}\}$. In the first case, $v \divideontimes [\Sigma, \Gamma \rhd \phi]$, and so $[\Sigma, \Gamma \rhd \phi] \notin \mathcal{A}_{\text{SF}}(V)$. In the second case, $v \divideontimes [\Gamma \rhd \sigma]$, and so $[\Gamma \rhd \sigma] \notin \mathcal{A}_{\text{SF}}(V)$.

RTL Set-Set: Suppose that A is monotonic and completely transitive, and $[\Gamma \succ \Delta] \notin A$. By complete transitivity (since $\mathcal{L} \subseteq \mathcal{L}$), there are some $\Lambda_1 \cup \Lambda_2 = \mathcal{L}$ such that $[\Lambda_1, \Gamma \succ \Delta, \Lambda_2] \notin A$. Let $\Theta_1 = \Lambda_1 \cup \Gamma$ and $\Theta_2 = \Lambda_2 \cup \Delta$, so $[\Theta_1 \succ \Theta_2]$ is this same argument. Since A is monotonic, there is no $b \sqsubseteq [\Theta_1 \succ \Theta_2]$ such that $b \in A$. Let v be the exact counterexample to $[\Theta_1 \succ \Theta_2]$. Note that since $\Theta_1 \cup \Theta_2 = \mathcal{L}$, we have $v \in \mathfrak{V}_3^t$.[12] For any argument a, $v \divideontimes a$ iff $a \sqsubseteq [\Theta_1 \succ \Theta_2]$. And we know there is no such $a \in A$, so $v \in \mathcal{V}_3^t(A)$. But $v \divideontimes [\Gamma \succ \Delta]$, so $[\Gamma \succ \Delta] \notin \mathcal{A}(\mathcal{V}_3^t(A))$.[13]

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[12]Refer to the proof of fact 2.

[13]Unlike the $\mathcal{V}_3^r$ case, here it would in general be false to claim that $\mathcal{V}_4(A) = \mathcal{V}_3^t(A)$. The trouble is that for any $v \in \mathfrak{V}_3^t$, there is some $w \notin \mathfrak{V}_3^t$ such that $w \sqsubseteq v$. But as $\mathcal{V}_4(A)$ is always closed downwards along $\sqsubseteq$ (which we will see later), this means that the only sets A for which $\mathcal{V}_4(A) = \mathcal{V}_3^t(A)$ are those for which these sets of valuations are both empty. This in turn is the case iff $[\succ] \in A$, since every valuation is a counterexample to $[\succ]$, and

SET-FMLA: similar, but involving a bit more fuss. Suppose that A is monotonic and completely transitive, and $[\Gamma \rhd \phi] \notin A$; we want some $v \in \mathcal{V}_3^t(A)$ with $v ※ [\Gamma \rhd \phi]$. It is determined as follows:

- $v(\psi) = \top$ iff $[\Gamma \rhd \psi] \in A$,
- $v(\psi) = \mathrm{I}$ iff $[\Gamma \rhd \psi] \notin A$ and $\psi \in \Gamma$, and
- $v(\psi) = \bot$ iff $[\Gamma \rhd \psi] \notin A$ and $\psi \notin \Gamma$.

Clearly $v \in \mathfrak{V}_3^t$. Moreover, $v ※ [\Gamma \rhd \phi]$; $v[\Gamma] \subseteq \{\top, \mathrm{I}\}$ by the first two bullet points and $v(\phi) \in \{\bot, \mathrm{I}\}$ by the last two.

It remains to show that there is no $[\Gamma' \rhd \psi] \in A$ with $v ※ [\Gamma' \rhd \psi]$. So suppose $v ※ [\Gamma' \rhd \psi]$, to show $[\Gamma' \rhd \psi] \notin A$. Let $\Theta = \Gamma' \setminus \Gamma$. By the way v was constructed, we must have $[\Gamma \rhd \psi] \notin A$, and for each $\theta \in \Theta$, $[\Gamma \rhd \theta] \in A$. So by the complete transitivity of A, $[\Theta, \Gamma \rhd \psi] \notin A$. But since $\Theta \cup \Gamma = \Gamma' \cup \Gamma$, this means $[\Gamma', \Gamma \rhd \psi] \notin A$, and thus by monotonicity $[\Gamma' \rhd \psi] \notin A$.

$\mathcal{V}_2$ This case is well-known. See [50], Theorem 1.1 for the SET-FMLA case and Theorem 2.1 for the SET-SET case. ■

This leads us to an expanded version of what is known as 'Suszko's thesis'. Say that a set V of valuations is a *presentation* of a set A of arguments iff $A = \mathcal{A}(V)$. Then one basic form of Suszko's thesis is that every monotonic, reflexive, and completely transitive set of arguments has a bivaluational presentation (a presentation $V \subseteq \mathfrak{V}_2$). We have the materials here to go farther:

COROLLARY 1. *(Suszko)*

- *Every monotonic set of arguments has a presentation $V \subseteq \mathfrak{V}_4$.*
- *Every monotonic and reflexive set of arguments has a presentation $V \subseteq \mathfrak{V}_3^r$.*
- *Every monotonic and completely transitive set of arguments has a presentation $V \subseteq \mathfrak{V}_3^t$.*
- *Every monotonic, reflexive and completely transitive set of arguments has a presentation $V \subseteq \mathfrak{V}_2$.*

Footnote 13 continued
v_* (the valuation that assigns $*$ to every sentence) is not a counterexample to any other argument (in either framework). Assuming monotonicity as we have, then, means that the claim would only be true when A is the set of *all* SET-SET arguments.

PROOF. Let A be a set of arguments satisfying the structural properties in question. By Theorem 2, $A = \mathcal{AV}(A)$, where $\mathcal{V}(A)$ picks out only valuations of an appropriate sort. ∎

Theorem 2 and Corollary 1 thus generalize their well-known fourth parts, dealing with bivaluations, to more general classes of valuations. For further discussion of the special cases involving bivaluations, see for example [13, 36, 38, 48, 54, 55], and especially [49].[14]

Beyond the bivaluational case, these facts are are less well-studied. Chemla et al. [14] is an interesting recent look at Suszko's thesis that pushes beyond the bivaluational case in a different kind of way than we have done here. See [32, Prop. 2] for our SET-SET tetravaluational case, and the discussion there following this proposition for claims of the two SET-SET trivaluational cases, including a comment covering much the same point as our footnote 13. (The presentation there is different and slightly more general, involving two languages rather than one, but the claims are essentially the same when taken in their one-language special case.) The SET-SET trivaluational claims are repeated, also without proof, in [46, fn. 13]. As far we know, this is the first published proof of the SET-SET trivaluational claims. We also believe this is the first statement of the SET-FMLA tetravaluational claim, and of the SET-FMLA trivaluational claims.

There are results related to (the RTL directions of) the trivaluational claims in [37, Thm. 3.2(i)] for SET-FMLA-$\mathfrak{V}_3^t$ and in [23, Thm. 5] for SET-FMLA-$\mathfrak{V}_3^f$. Those ideas are further explored in [11]. However, those results do not involve a Galois connection at all, but instead exploit different connections between valuations and arguments. The relation between that approach and our own is interesting, and not completely straightforward; we discuss the situation in [26], but here simply pass over it without further comment.

2.2. Closed Sets of Valuations

It's now time to return to the bilattice structure of $\mathfrak{V}_4$. Each of $\sqsubseteq$ and $\preccurlyeq$ is a complete lattice order on $\mathfrak{V}_4$; every set V of tetravaluations has an *information meet* $\bigsqcap V$, an *information join* $\bigsqcup V$, a *truth meet* $\bigcurlywedge V$, and a *truth join* $\bigcurlyvee V$. Moreover, this gives what Fitting [22] calls an *interlaced*

[14] Wansing and Shramko [56, Theorem 4.1] contradicts this fourth part, claiming (in effect) that in the SET-FMLA case it suffices that the set contain all arguments of the form $[\Gamma, A \rhd A]$. (This is a stronger constraint than reflexivity, but weaker than reflexivity plus monotonicity, and it makes no provision at all for complete transitivity.) This error is corrected in [57].

bilattice: each pair of lattice operators preserves not only the corresponding order (as any lattice operators must), but also the *other* order.[15]

As we briefly noted in Section 1.1, this bilattice structure does not extend to our smaller background sets of valuations: $\mathfrak{V}_3^r$ is not closed under $\bigsqcup$ (since $\bot \sqcup \top = \mathrm{I}$); $\mathfrak{V}_3^t$ is not closed under $\bigsqcap$ (since $\bot \sqcap \top = *$); and $\mathfrak{V}_2$ is not closed under either $\bigsqcup$ or $\bigsqcap$. All these sets are closed under both $\curlywedge$ and $\curlyvee$, however; and for now, it is $\curlywedge$ that is of most concern.

THEOREM 3. *A set V of valuations (considered as part of any of our four possible $\mathfrak{V}$, so long as $V \subseteq \mathfrak{V}$) is:*

- $\mathcal{VA}_{\text{SS}}$*-closed iff it is closed downwards along $\sqsubseteq$, and*
- $\mathcal{VA}_{\text{SF}}$*-closed iff it is closed downwards along $\sqsubseteq$ and closed under $\curlywedge$s.*

PROOF. Recall that V is closed iff $V = \mathcal{VA}(V)$. In each case, we proceed by showing first that if $V = \mathcal{V}(A)$ for *any* set A of arguments in the appropriate framework, then it has the properties in question, and showing second that if V has the needed properties, then $\mathcal{VA}(V) \subseteq V$. (For any set V of valuations, we already have $V \subseteq \mathcal{VA}(V)$ by Theorem 1.)

$\mathcal{VA}_{\text{SS}}$

LTR Suppose $V = \mathcal{V}(A)$ for some $A \subseteq \mathfrak{A}_{\text{SS}}$, to show that V is closed downwards along $\sqsubseteq$. Let w be a valuation of the appropriate kind such that $w \notin V$ and $w \sqsubseteq v$. Since $w \notin \mathcal{V}(A)$ and w is of the appropriate kind, there must be some $a \in A$ with $w \divideontimes a$. By Fact 1, $v \divideontimes a$ as well. So $v \notin \mathcal{V}(A)$.

RTL Suppose V is closed downwards along $\sqsubseteq$. Take any v of the appropriate kind such that $v \notin V$, to show $v \notin \mathcal{VA}_{\text{SS}}(V)$. Consider the argument a exactly counterexampled by v. For any w, $w \divideontimes a$ iff $v \sqsubseteq w$. As $v \notin V$, v is of the appropriate kind, and V is closed downwards along $\sqsubseteq$, no such w is in V. But then $a \in \mathcal{A}_{\text{SS}}(V)$. Since $v \divideontimes a$, then, $v \notin \mathcal{VA}_{\text{SS}}(V)$.

$\mathcal{VA}_{\text{SF}}$

LTR Suppose $V = \mathcal{V}(A)$ for some $A \subseteq \mathfrak{A}_{\text{SF}}$, to show that V is closed downwards along $\sqsubseteq$ and closed under $\curlywedge$. Showing that V is closed downwards along $\sqsubseteq$ is exactly the same as in the SET-SET case, since Fact 1 applies equally to both frameworks.

[15] Fitting [22, p. 96] points out that our bilattice on values is interlaced; it follows from this and Proposition 3.2 there that our bilattice on valuations is also interlaced.

Suppose, then, that V is not closed under $\curlywedge$s; that there is some w of the appropriate kind such that $w = \curlywedge_{i \in I} v_i$ with $v_i \in \mathcal{V}(A)$ for all $i \in I$, but $w \notin \mathcal{V}(A)$. Since $w \notin \mathcal{V}(A)$, but w is of the appropriate kind, there must be some $[\Gamma \rhd \phi] \in A$ with $w ※ [\Gamma \rhd \phi]$. And since $v_i \in \mathcal{V}(A)$ for all $i \in I$, there is no $i \in I$ with $v_i ※ [\Gamma \rhd \phi]$. This is to say that for every $i \in I$, either there is some $\gamma \in \Gamma$ with $v_i(\gamma) \in \{\bot, *\}$ or else $v_i(\phi) \in \{\top, *\}$.

Since $w ※ [\Gamma \rhd \phi]$, it must be that $w(\phi) \in \{\bot, \text{I}\}$; if $v_i(\phi) \in \{\top, *\}$ for every $i \in I$ this is not possible, since $\{\top, *\}$ is closed under $\curlywedge$. So at least some $i \in I$ gives $v_i(\gamma) \in \{\bot, *\}$ for some $\gamma \in \Gamma$. But then $w(\gamma) \in \{\bot, *\}$ as well, and so w cannot be a counterexample to a. Contradiction.

RTL Suppose V is closed downwards along $\sqsubseteq$ and closed under $\curlywedge$s. Take any $w \in \mathcal{V}\mathcal{A}_{\text{SF}}(V)$, to show $w \in V$. Let $\Gamma = \{\psi : w(\psi) \in \{\top, \text{I}\}\}$, and consider all the ϕ such that $w ※ [\Gamma \rhd \phi]$; let these be $\{\phi_i\}_{i \in I}$. Since $w \in \mathcal{V}\mathcal{A}_{\text{SF}}(V)$, all such $[\Gamma \rhd \phi_i] \notin \mathcal{A}_{\text{SF}}(V)$, so for each $i \in I$ there is some $v_i \in V$ such that $v_i ※ [\Gamma \rhd \phi_i]$. We claim that $w \sqsubseteq \curlywedge_{i \in I} v_i$; from this it follows that $w \in V$, since V is closed under $\curlywedge$ and downwards along $\sqsubseteq$.

To see this, consider any $\psi \in \mathcal{L}$, to show that $w(\psi) \sqsubseteq \curlywedge_{i \in I} v_i(\psi)$. There are four cases, depending on $w(\psi)$:

- $w(\psi) = \top$. Then $\psi \in \Gamma$, so $v_j(\psi) \in \{\top, \text{I}\}$ for each $j \in I$, and so $\curlywedge_{i \in I} v_i(\psi) \in \{\top, \text{I}\}$.
- $w(\psi) = \text{I}$. Then again $\psi \in \Gamma$, so $v_j(\psi) \in \{\top, \text{I}\}$ for each $j \in I$, and so $\curlywedge_{i \in I} v_i(\psi) \in \{\top, \text{I}\}$. But now $w ※ [\Gamma \rhd \psi]$, so ψ is ϕ_j for some $j \in I$, and so $v_j(\psi) = \text{I}$. Thus, $\curlywedge_{i \in I} v_i(\psi) = \text{I}$.
- $w(\psi) = \bot$. In this case, $w ※ [\Gamma \rhd \psi]$, so $\psi = \phi_j$ for some j. But then $v_j(\psi) \in \{\bot, \text{I}\}$.
- $w(\psi) = *$. Since $* \sqsubseteq \bullet$ for all values $\bullet$, in this case we're done.

■

Theorem 3 gives us a way to see which sets of valuations are closed. As with Theorem 2, the bivaluational case is known—but it is less well-known. See [19, p. 202] for a helpful 'Historical Remark' on the bivaluational Set-Set case, which is to be found here and there in the literature. Less well-known still is the bivaluational Set-Fmla case; but see [30]; [33, Thm. 0.2.5]; [34, Thm. 1.14.9], as well as the works cited at [34, p. 101]. Importantly, when v, w are bivaluations, $v \sqsubseteq w$ iff $v = w$; this means that *every* set of bivaluations is $\mathcal{V}_2\mathcal{A}_{\text{SS}}$ closed. As a result, these facts often sound very different when discussion is restricted to bivaluations (as it almost always is). There is no need at all in such settings to think about the information order.

For the tetravaluational SET-SET case, see [46, Fact 8]. Compare also [37, Thms. 3.2(ii) and 4.1], which contain facts reminiscent of the SET-FMLA-$\mathfrak{V}_3^t$ case here. (Malinowski's $\Cap$ is exactly our $\curlywedge$ (restricted to transitive trivaluations), but he does not address the question of closure in our sense.)

Digression: Other Frameworks. We pause at this point to briefly mention some counterparts of the above results for less well studied logical frameworks—in particular FMLA-SET and FMLA-FMLA. (We take up the notation $\mathcal{A}_{\text{FS}}, \mathcal{A}_{\text{FF}}, \mathfrak{A}_{\text{FS}}$, and $\mathfrak{A}_{\text{FF}}$, all with the obvious meanings, in this digression.) Our primary reason to discuss these frameworks is to bring out certain symmetries in the above results, illustrating the connections between frameworks and the closure of classes of valuations under truth meets and truth joins. That being said, these frameworks have recieved a limited amount of attention in the literature. Interest in the FMLA-FMLA framework has come from work connected to algebraic logic (see, for example, [34, pp. 246–248] for a discussion of the logic of distributive lattices, and [29] for a discussion of the logic of orthomodular lattices), as well as work connected to sublogics of intuitionistic logic such as the basic propositional logic of [1].[16] The FMLA-SET framework has primarily been studied in connection to vagueness, where it is used to bring out certain structural features of subvaluational approaches—concerning which the interested reader should consult [15] for a survey of some of the logical developments.

The bivaluational case of Theorem 4 is mentioned in passing in [35, fn. 7]; we know of no discussion of the other cases of this result, or of Theorem 5.

THEOREM 4. *A set V of valuations (considered as part of any of our four possible $\mathfrak{V}$, so long as $V \subseteq \mathfrak{V}$) is $\mathcal{V}\mathcal{A}_{\text{FS}}$-closed iff it is closed downwards along $\sqsubseteq$ and closed under $\curlyvee$s.*

PROOF. As the case of SET-FMLA and SET-SET above, we proceed by showing first that if $V = \mathcal{V}(A)$ for any set of FMLA-SET arguments A, then it is closed downwards along $\sqsubseteq$ and closed under $\curlyvee$, and secondly that if V is closed under these properties then $\mathcal{V}(\mathcal{A}_{\text{FS}}(V)) \subseteq V$.

LTR Suppose $V = \mathcal{V}(A)$ for some $A \subseteq \mathfrak{A}_{\text{FS}}$, to show that V is closed downwards along $\sqsubseteq$ and closed under $\curlyvee$s. That V is closed downwards along $\sqsubseteq$ is exactly as in the SET-SET and SET-FMLA cases above.

[16] Dunn and Hardegree [19, pp. 191–194] is a brief introductory discussion of the FMLA-FMLA framework.

Suppose, then, that V is not closed under $\curlyvee$s. Then there is some w of the appropriate kind such that $w = \curlyvee_{i \in I} v_i$ with $v_i \in \mathcal{V}(A)$ for all $i \in I$, but that $w \notin \mathcal{V}(A)$. Since $w \notin \mathcal{V}(A)$, but w is of the appropriate kind, there must be some $[\phi \rhd \Delta] \in A$ with $w \divideontimes [\phi \rhd \Delta]$. And since $v_i \in \mathcal{V}(A)$ for all $i \in I$, there is no $i \in I$ with $v_i \divideontimes [\phi \rhd \Delta]$. That is to say, for every $i \in I$, either $v_i(\phi) \in \{\bot, *\}$, or else for all $\delta \in \Delta$ we have $v_i(\delta) \in \{\top, *\}$.

Since $w \divideontimes [\phi \rhd \Delta]$, it must be that $w(\phi) \in \{\top, \mathbb{I}\}$; if $v_i(\phi) \in \{\bot, *\}$ for every $i \in I$ this is not possible, since $\{\bot, *\}$ is closed under $\curlyvee$. So at least some $i \in I$ gives $v_i(\delta) \in \{\top, *\}$ for some $\delta \in \Delta$. But then $w(\delta) \in \{\top, *\}$ as well, and so w cannot be a counterexample to $[\phi \rhd \Delta]$. Contradiction.

RTL Suppose that V is closed downwards along $\sqsubseteq$ and closed under $\curlyvee$s. Take any $w \in \mathcal{V}\mathcal{A}_{\text{FS}}(V)$, to show $w \in V$. Let $\Delta = \{\psi : w(\psi) \in \{\bot, \mathbb{I}\}\}$, and consider all the ϕ such that $w \divideontimes [\phi \rhd \Delta]$; let these be $\{\phi_i\}_{i \in I}$. Since $w \in \mathcal{V}\mathcal{A}_{\text{FS}}(V)$, all such $[\phi \rhd \Delta] \notin \mathcal{A}_{\text{FS}}(V)$, so for each $i \in I$ there is some $v_i \in V$ such that $v_i \divideontimes [\phi_i \rhd \Delta]$. We claim $w \sqsubseteq \curlyvee_{i \in I} v_i$; from this it follows that $w \in V$, since V is closed under $\curlyvee$ and downwards along $\sqsubseteq$.

To see this, consider any $\psi \in \mathcal{L}$, to show that $w(\psi) \sqsubseteq \curlyvee_{i \in I} v_i(\psi)$. There are four cases, depending on $w(\psi)$:

- $w(\psi) = \top$. Then we have $w \divideontimes [\psi \rhd \Delta]$, so $\psi = \phi_i$ for some $j \in I$, and so $\curlywedge_{i \in I} v_i(\psi) \in \{\top, \mathbb{I}\}$.
- $w(\psi) = \mathbb{I}$. Then again we have $w \divideontimes [\psi \rhd \Delta]$, so $\psi = \phi_i$ for some $j \in I$, and so $\curlywedge_{i \in I} v_i(\psi) \in \{\top, \mathbb{I}\}$. As $w(\psi) = \mathbb{I}$, though, we also ahve $\psi \in \Delta$, and so $v_j(\psi) \in \{\bot, \mathbb{I}\}$ for all $j \in I$, and so $\curlywedge_{i \in I} v_i(\psi) = \mathbb{I}$.
- $w(\psi) = \bot$. In this case $\psi \in \Delta$, so $v_j(\psi) \in \{\bot, \mathbb{I}\}$ for all $j \in I$, and so $\curlywedge_{i \in I} v_i(\psi) \in \{\bot, \mathbb{I}\}$.
- $w(\psi) = *$. Since $* \sqsubseteq \bullet$ for all values $\bullet$, in this case we're done. ■

The situation is similar for the Fmla-Fmla framework:

Theorem 5. *A set V of valuations (considered as part of any of our four possible $\mathfrak{V}$, so long as $V \subseteq \mathfrak{V}$) is $\mathcal{V}\mathcal{A}_{\text{FF}}$-closed iff it is closed downwards along $\sqsubseteq$, closed under $\curlywedge$s, and closed under $\curlyvee$s.*

PROOF.

LTR Suppose that $V = \mathcal{V}(A)$ for some collection $A \subseteq \mathfrak{A}_{\text{FF}}$, to show that V is closed downwards along $\sqsubseteq$, and closed under $\curlywedge$s and $\curlyvee$s. That V is closed downwards along $\sqsubseteq$ is just as before.

To see that $\mathcal{V}(A)$ is closed under $\curlyvee$s, suppose $w = \curlyvee_{i \in I} v_i$ and $w \notin \mathcal{V}(A)$ (with w and all v_i of the appropriate kind), to show that there is some $v_i \notin \mathcal{V}(A)$. Since $w \notin \mathcal{V}(A)$, but w is of the appropriate kind, there must be some $[\phi \succ \psi] \in A$ with $w ※ [\phi \succ \psi]$. As $w ※ [\phi \succ \psi]$, it must be that $w(\phi) \in \{\top, \mathbb{I}\}$, and so there is some v_i with $v_i(\phi) \in \{\top, \mathbb{I}\}$; let this be v_j. But also as $w ※ [\phi \succ \psi]$, it must be that $w(\psi) \in \{\bot, \mathbb{I}\}$, and so there must be no v_i with $v_i(\psi) \in \{\top, *\}$; in particular, $v_j(\psi) \notin \{\top, *\}$. Thus, $v_j(\psi) \in \{\bot, \mathbb{I}\}$, and so $v_j ※ [\phi \succ \psi]$. Thus $v_j \notin \mathcal{V}(A)$.

To see that $\mathcal{V}(A)$ is closed under $\curlywedge$s, repeat the previous paragraph while standing on your (truth) head.

RTL Suppose that V meets the needed closure conditions, and take some $w \in \mathcal{V}\mathcal{A}_{\text{FF}}(V)$, to show $w \in V$.

Let $\{\phi_i\}_{i \in I} = \{\phi \mid w(\phi) \in \{\top, \mathbb{I}\}\}$, and let $\{\psi_j\}_{j \in J} = \{\psi \mid w(\psi) \in \{\bot, \mathbb{I}\}\}$. Since $w ※ [\phi_i \succ \psi_j]$ for each $i \in I, j \in J$, we know that no such argument is in $\mathcal{A}_{\text{FF}}(V)$; thus, for each $i \in I, j \in J$ there is some $v_{ij} \in V$ with $v_{ij} ※ [\phi_i \succ \psi_j]$.

We claim that $w \sqsubseteq \curlyvee_{i \in I} \curlywedge_{j \in J} v_{ij}$; it follows that $w \in V$ by the closure conditions. To verify the claim, consider any $\chi \in \mathcal{L}$; we show that $w(\chi) \sqsubseteq \left[\curlyvee_{i \in I} \curlywedge_{j \in J} v_{ij}\right](\chi)$. Here there are four cases:

- $w(\chi) = \top$. Then $\chi = \phi_k$ for some $k \in I$. As each $v_{kj} ※ [\chi \succ \psi_j]$ it follows that $v_{kj}(\chi) \in \{\top, \mathbb{I}\}$ for each $j \in J$; so $\left[\curlywedge_{j \in J} v_{kj}\right](\chi) \in \{\top, \mathbb{I}\}$; and so $\left[\curlyvee_{i \in I} \curlywedge_{j \in J} v_{ij}\right](\chi) \in \{\top, \mathbb{I}\}$ as well.
- $w(\chi) = \bot$. Then $\chi = \psi_k$ for some $k \in J$. As each $v_{ik} ※ [\phi_i \succ \chi]$ it follows that $v_{ik}(\chi) \in \{\bot, \mathbb{I}\}$ for each $i \in I$; so $\left[\curlywedge_{j \in J} v_{ij}\right](\chi) \in \{\bot, \mathbb{I}\}$ for each $i \in I$, and so $\left[\curlyvee_{i \in I} \curlywedge_{j \in J} v_{ij}\right](\chi) \in \{\bot, \mathbb{I}\}$ as well.
- $w(\chi) = \mathbb{I}$. Then $\chi = \phi_k$ for some k, and $\chi = \psi_l$ for some l. Thus, for all $j \in J$, $v_{kj}(\chi) \in \{\top, \mathbb{I}\}$, and in particular $v_{kl}(\chi) = \mathbb{I}$. As a result, $\left[\curlywedge_{j \in J} v_{kj}\right](\chi) = \mathbb{I}$. Moreover, for every $i \in I$, we must

have $v_{il}(\chi) \in \{\bot, \top\}$, and so $\left[\bigcurlywedge_{j\in J} v_{ij}\right](\chi) \in \{\bot, \top\}$ for every $i \in I$. As a result, $\left[\bigcurlyvee_{i\in I} \bigcurlywedge_{j\in J} v_{ij}\right](\chi) = \top$.

- $w(\chi) = *$. Since $* \sqsubseteq \bullet$ for all values $\bullet$, in this case we're done. ■

2.3. Lattices of Closed Sets

So far, we've got eight Galois connections in play. Each of these, simply by being a Galois connection, restricts to an order-inverting order-isomorphism between *closed* sets, considered as ordered by $\subseteq$. This section explores these eight orders.

2.3.1. Complete Lattices The first thing to note is that all eight are complete lattice orders.[17] Given any closed sets $\{A_i\}_{i\in I}$ of arguments, $\bigcap_{i\in I} A_i$ is also closed, and is clearly the greatest lower bound of the A_is.

Least upper bounds also exist, but the situation is slightly more complex. For any closed sets $\{A_i\}_{i\in I}$ of arguments, $\bigcup_{i\in I} A_i$ exists, but it need not always be closed. Here the situation depends on which Galois connection we are considering.

So long as the set of valuations in play is $\mathfrak{V}_4$, unions of closed sets of arguments are indeed always closed. This is because closed sets of arguments need obey only monotonicity, and this is preserved by unions, essentially because monotonicity is a one-premise rule. However, when we consider narrower sets of valuations, we impose more restrictions on closed sets of arguments, and sets of arguments meeting these restrictions need not be closed under unions. For the case of $\mathfrak{V}_3^r$, this is perhaps not obvious, but consider the empty union: it is the empty set of arguments, and this is certainly not reflexive.[18] (The reflexive sets of arguments are, however, closed under *nonempty* unions, essentially because reflexivity is a zero-premise rule.)

So to arrive at least upper bounds, simply taking the union of closed sets of arguments does not suffice. We need instead, for closed sets $\{A_i\}_{i\in I}$, to consider $\mathcal{AV}(\bigcup_{i\in I} A_i)$. In the cases where $\bigcup_{i\in I} A_i$ is already closed, this makes no difference; but in other cases it does. Anyhow, it is this *closed*

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[17] Wójcicki [58, Corollary 1.5.4] points this out for the case of Set-Fmla arguments and bivaluations.

[18] This contradicts the final claim in the statement of [24, Fact 2.1], which concerns itself with the lattice of reflexive and monotonic sets of Set-Fmla arguments—which is to say, Set-Fmla-$\mathfrak{V}_3^r$. This claims that the least upper bound of a set of such sets is its union; but this is only true for the *nonempty* sets of such sets. The least upper bound of the empty set of such sets is the smallest reflexive set, not the empty set.

union that gives least upper bounds in any of our eight orders. (This follows directly from properties of $\bigcup$ together with the fact that $\mathcal{AV}$ is a closure.)

Since the eight orders on closed sets of arguments are all complete lattices, so too are the eight orders on closed sets of valuations. After all, these are isomorphic! But it is useful to check in anyhow, to see where meets and joins among closed sets of valuations do and do not coincide with set-theoretic intersections and unions.

Here, the question turns on which framework is in play. If we are working in Set-Set, then the closed sets of valuations (which need only be closed downwards along $\sqsubseteq$) are closed under both intersections and unions, essentially because downward closure is a one-premise rule.

But if we are working in Set-Fmla, this need not be the case: the additional requirement that closed sets be closed under $\bigcurlywedge$s can matter. While intersections of collections of closed sets of valuations are indeed always closed, unions need not be. Let $v_\top$ be the valuation that assigns a $\top$ to every formula; this is in $\mathfrak{V}_2$, and so is in play no matter which background set of valuations we work with. Since $\bigcurlywedge \emptyset = v_\top$, when we work in the Set-Fmla framework $v_\top$ is in every closed set of valuations. But $\bigcup \emptyset = \emptyset$; the union of the empty set of closed sets of valuations is not closed. So to arrive at least upper bounds in the $\subseteq$ order on closed sets of valuations, we need in general to consider $\mathcal{VA}(\bigcup)$, not just $\bigcup$.

2.3.2. Distributivity This is enough to see that five of our eight lattices are distributive. First, any of the four lattices involving the Set-Set framework must be completely distributive, since on the valuational side their meets and joins are simply set-theoretic intersections and unions, and these are indeed completely distributive. Second, either of the two lattices involving the full set $\mathfrak{V}_4$ of tetravaluations must be completely distributive, since on the argument side their meets and joins are simply set-theoretic intersections and unions. Since one of our lattices involves both the Set-Set framework and $\mathfrak{V}_4$, this gives us four plus two is five.

What about the other three? These all involve the Set-Fmla framework, and concern its connections to $\mathfrak{V}_3^r$, $\mathfrak{V}_3^t$, and $\mathfrak{V}_2$. For one of these, we again have complete distributivity.[19]

Fact 3. *The lattice of closed sets formed by the* Set-Fmla *framework and* $\mathfrak{V}_3^r$ *is completely distributive.*

[19] See also [24, Fact 2.1] for completeness and distributivity—but not for complete distributivity—of this case. Note, though, that this is the Fact containing the error explained in footnote 18.

PROOF. Let $\boldsymbol{R} = \{[\phi \rhd \phi] | \phi \in \mathcal{L}\}$, and call a set $\boldsymbol{B}$ of arguments *irreflexive* iff $\boldsymbol{B} \cap \boldsymbol{R} = \emptyset$. Call a set $\boldsymbol{B}$ of arguments *partially monotonic* iff $\boldsymbol{B} \cup \boldsymbol{R}$ is monotonic. That is, a partially monotonic set is basically monotonic, except it does not need to contain the arguments in $\boldsymbol{R}$, even if monotonicity would require them.

The set of irreflexive and partially monotonic sets of arguments is not of much independent interest (which is why these terms are defined only internally to this proof), but it *is* isomorphic to the set of closed sets of arguments. Each closed set $\boldsymbol{A}$ determines an irreflexive and partially monotonic set $\boldsymbol{A} \setminus \boldsymbol{R}$, and each irreflexive and partially monotonic set $\boldsymbol{B}$ determines a closed set $\boldsymbol{B} \cup \boldsymbol{R}$. Moreover, these determinations are mutually inverse and order-preserving. (For them to be mutually inverse, it is important that $\boldsymbol{R} \subseteq \boldsymbol{A}$ for each closed $\boldsymbol{A}$.)

The set of irreflexive and partially monotonic sets of arguments, moreover, *is* closed under arbitrary unions and intersections. For example, the empty union determines the empty set of arguments, and the empty set is indeed irreflexive and partially monotonic. (Indeed, it is monotonic.) So $\subseteq$ on this set forms a completely distributive lattice order. But we have already seen that this set is isomorphic to the set of closed sets of arguments; so that lattice too is completely distributive. ∎

This leaves $\mathfrak{V}_3^t$ and $\mathfrak{V}_2$. By contrast with the other six cases, these are not even distributive, let alone completely so, at least if our language has a reasonable size.

FACT 4. *If $|\mathcal{L}| \geq 3$, the lattices of closed sets formed by the* SET-FMLA *framework and $\mathfrak{V}_3^t$ or $\mathfrak{V}_2$ are not distributive.*

PROOF. In what follows, we show that the well-known lattice $\boldsymbol{M}_3$ is a sublattice of each of these lattices; this suffices for nondistributivity.[20]

Let $\boldsymbol{A}$ be the smallest set of arguments that is reflexive, monotonic, and completely transitive. (That is, $[\Gamma \rhd \phi] \in \boldsymbol{A}$ iff $\phi \in \Gamma$.) Consider three distinct formulas ψ_i for $i \in \{0, 1, 2\}$.[21] For the remainder of the proof, let $2 + 1 = 0$, for convenience.[22] For $i \in \{0, 1, 2\}$, let $\boldsymbol{A}_i$ be $\boldsymbol{A} \cup \{[\Gamma \rhd \psi_i] \mid \Gamma \subseteq \mathcal{L}\} \cup \{[\Gamma, \psi_{i+1} \rhd \phi] \mid \Gamma \cup \{\phi\} \subseteq \mathcal{L}\}$. That is, $\boldsymbol{A}_i$ includes the arguments in $\boldsymbol{A}$, plus all arguments with ψ_i as their conclusion, plus all arguments with ψ_{i+1} among their premises.

[20] For what $\boldsymbol{M}_3$ is and why this works, see [16, Ch. 4].

[21] Note that this is the first place $\mathcal{L}$'s cardinality has mattered, for any of our results.

[22] All we really need is some permutation on $\{\psi_0, \psi_1, \psi_2\}$; we will use $+1$ to indicate it.

Each such A_i is closed. Reflexivity and monotonicity are immediate. For complete transitivity, suppose $[\Gamma \rhd \sigma] \in A_i$ for each $\sigma \in \Sigma$ and $[\Sigma, \Gamma \rhd \phi] \in A_i$, to show $[\Gamma \rhd \phi] \in A_i$. If $\psi_{i+1} \in \Gamma$ or $\phi = \psi_i$, then $[\Gamma \rhd \phi] \in A_i$ directly, so we can assume neither of these is the case. Now, since for each $\sigma \in \Sigma$ we have $[\Gamma \rhd \sigma] \in A_i$, and since $\psi_{i+1} \notin \Gamma$, we have $\Sigma \subseteq \Gamma \cup \{\psi_i\}$. Thus, since $[\Sigma, \Gamma \rhd \phi] \in A_i$, also $[\Gamma, \psi_i \rhd \phi] \in A_i$. But this argument does not have ψ_{i+1} among its premises, so we must have $\phi \in \Gamma \cup \{\psi_i\}$. Since we have $\phi \neq \psi_i$, this gives $\phi \in \Gamma$. So $[\Gamma \rhd \phi] \in A_i$.

Let $i \neq j \in \{0, 1, 2\}$. Then $A_i \not\subseteq A_j$. Moreover, $A_i \cap A_j = A$; any argument added to A in any A_i is not added by either of the others. Finally, $\mathcal{AV}(A_i \cup A_j)$ is $\mathfrak{A}_{\text{SF}}$. This is because for any such i, j, either $i = j+1$ or $j = i+1$. Wlog, let it be the former. Then for any Γ, ϕ, we have $[\Gamma \rhd \psi_j] \in A_j$ and $[\Gamma, \psi_j \rhd \phi] \in A_i$. Closing $A_i \cup A_j$ under complete transitivity, then, gives $[\Gamma \rhd \phi] \in \mathcal{AV}(A_i \cup A_j)$.

The facts adduced in the last paragraph, though, mean that the lattice in question contains three incomparable elements (the A_is) any two of which have a common meet (A) and any two of which have a common join ($\mathfrak{A}_{\text{SF}}$). This is the nondistributive M_3, here found as a sublattice of our lattices, which are thereby themselves nondistributive. ■

3. Adding to the Language

In this final section, we consider the effects of adding connectives to the language. We use the bilattice structure present in our valuations to provide these connectives. With $\mathcal{L}$ and $\mathfrak{V}_4$ as before, we form the extended language $\mathcal{L}_+$. This treats formulas in $\mathcal{L}$ as 'atoms', and constructs complex formulas by way of new connectives. $\mathcal{L}_+$ has the full complement of of zeroary connectives $\top, \bot, \top\!\!\!\bot, *$, unary connectives $\sim$ and $\neg$, and binary connectives $\curlywedge, \curlyvee, \sqcap, \sqcup$; it is the *full language*. From here, we use ϕ, ψ, etc, for arbitrary formulas in $\mathcal{L}_+$.

We assume all valuations treat all these connectives appropriately. That is, for any v, $v(\top\!\!\!\bot) = \top\!\!\!\bot$, $v(\phi \curlywedge \psi) = v(\phi) \curlywedge v(\psi)$, and so on. The only connectives requiring remark here are the two negations. Truth negation $\sim$ swaps $\top$ and $\bot$ while leaving $\top\!\!\!\bot$ and $*$ fixed; and information negation $\neg$ does the opposite, swapping $\top\!\!\!\bot$ and $*$ while leaving $\top$ and $\bot$ fixed.[23]

[23] Information negation is called 'conflation' in [21,22]. This is an unfortunate name: it is a unary operation, but one conflates *multiple* things. (Indeed, it is $\sqcap$ that is most closely related to conflation, at least if [46] is broadly on the right track.) So we avoid Fitting's terminology here.

In exploring $\mathcal{L}_+$, we consider only sets of tetravaluations, rather than maintaining our four distinct background sets of valuations. This is because the constants $\top$ and $*$ force all valuations to use the values $\top$ and $*$. Even without these constants, we have $\top \sqcap \bot = *$ and $\top \sqcup \bot = \top$, so even bivaluations end up using all four values. Restricting ourselves to $\mathfrak{V}_2$, $\mathfrak{V}_3^r$, or $\mathfrak{V}_3^t$, then, would amount to imposing special requirements on formulas in $\mathcal{L}$ not obeyed by the remainder of the language.

3.1. (Information) Monotonicity

In the presence of these additional connectives, it is no longer a matter of definition that whenever $u \sqsubseteq v$, then $u(\phi) \sqsubseteq v(\phi)$ for every formula ϕ. The former claim is defined as before, by looking at all formulas in $\mathcal{L}$. But we have new formulas now to be checked in the latter claim. The claim now holds only in a restricted form; $\neg$ causes trouble for it.

FACT 5. *If $u \sqsubseteq v$, then $u(\phi) \sqsubseteq v(\phi)$ for any formula ϕ not containing $\neg$.*

PROOF. For atoms, the result follows by definition. For the inductive step, the constants (whichever are present) are immediate; it remains to check unary and binary connectives. Here, though, all of our connectives but $\neg$ preserve the $\sqsubseteq$ order; $\sim$ directly, $\sqcup$ and $\sqcap$ as lattice operations in the $\sqsubseteq$ lattice, and $\curlyvee$ and $\curlywedge$ because our bilattice is interlaced. ∎

Because Fact 5 does not hold in full generality, we do not have for $\mathcal{L}_+$ a fact corresponding to Fact 1. This turns out to create complications in the coming sequent calculus for $\mathcal{L}_+$. Moreover, if we think of $\sqsubseteq$ as really telling us something about *information*, then operations that fail to preserve it can easily seem nonsensical.[24] So we also consider $\mathcal{L}_{\sqsubseteq}$, which contains all the connectives of $\mathcal{L}_+$ except $\neg$. Since every connective of $\mathcal{L}_{\sqsubseteq}$ is information monotonic, so is each of its formulas; the restriction in Fact 5 is no restriction at all for $\mathcal{L}_{\sqsubseteq}$, which does not contain $\neg$ in the first place.

For a language like this, these valuations are well-studied, both with the full complement of operations in $\mathcal{L}_+$ and with the restricted complement of operations in $\mathcal{L}_{\sqsubseteq}$. (It is perhaps even better-studied still if we ignore the information connectives entirely and stick only to the truth connectives; in this form it is studied, for example, in [5,6,17], [42, Ch. 3]. We do not consider this possibility here, though.) See, for example, [2,4,12,21,22,27]. In particular, it is worth noting that $\mathcal{L}_+$ is functionally complete: every

[24] For discussion, see [9,10,43].

operation on $\{\bot, \top, \text{I}, *\}$, of any arity, is definable in $\mathcal{L}_+$. Moreover, every *information monotonic* operation of any arity is definable in $\mathcal{L}_\sqsubseteq$.[25]

However, where these structures have been connected to consequence relations, it is typically in ways other than the way we have chosen here. For example, [2, Def. 3.2a] have (in our notation, not theirs) that $v \divideontimes [\Gamma \succ \Delta]$ iff $v[\Gamma] \subseteq \{\top, \text{I}\}$ and $v[\Delta] \subseteq \{\bot, *\}$. This definition comes apart from ours in having $\{\bot, *\}$ as the key counterexample values for conclusions, instead of our $\{\bot, \text{I}\}$. As a result, if $v(\phi) = v(\psi) = \text{I}$, then v is a counterexample in our sense but not theirs to $[\phi \succ \psi]$; and if $v(\phi) = \top$ and $v(\psi) = *$, then v is a counterexample in their sense but not ours. The same kind of thing happens for other well-known counterexample relations involving these and similar valuations. So while the valuations themselves are well-known, nonetheless the results to follow are novel (as far as we know), since they rely on the particular counterexample relation we study in this paper, which is not a usual one for these valuations.

3.2. Sequent Calculi

We present a systematic way to give a sequent system for $\mathcal{A}_{\text{SS}}(V)$ for $V \subseteq \mathfrak{V}_4$. (This covers corresponding questions about $\mathcal{A}_{\text{SF}}(V)$ for such V, since each Set-Fmla argument has a corresponding singleton-conclusion Set-Set argument.)

The objects manipulated in these calculi are the arguments themselves. We begin with a system just for $\mathcal{L}_\sqsubseteq$, given in Figure 2. The rules for the truth connectives in this system are familiar; they are typical rules for the corresponding connectives of classical logic. (Indeed, by taking $\mathfrak{V}_3^r$ as the background set V, fixing the initial sequents, the truth rules give exactly classical logic for the truth vocabulary.) The informational rules, however, are different, involving hybrids of more familiar classical rules. For example, $\sqcap$ has the left rule of $\curlywedge$ and the right rule of $\curlyvee$; while $\sqcup$ has the left rule of $\curlyvee$ and the right rule of $\curlywedge$. Continuing this pattern, $*$ has the left rule of $\bot$ and the right rule of $\top$; while I, in a kind of degenerate way, has the left rule of $\top$ and the right rule of $\bot$.[26]

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[25] For both results, see [39, pp. 49–50]. Thanks to José Martínez for providing this reference, as well as providing (p.c.) a different proof of the latter claim.

[26] Similar behaviour reveals itself in the proof system for $\mathcal{L}_\sqsubseteq$ given in [2, p. 37ff], but in a different way. Recall that their notion of v being a counterexample to $[\Gamma \succ \Delta]$, differs from ours in requiring $v[\Gamma] \subseteq \{\top, \text{I}\}$ and $v[\Delta] \subseteq \{\bot, *\}$. This leads to a range of differences in presentation. In particular, for their notion of validity the connectives need separate rules for when they occur on their own and when they occur embedded under $\sim$. (This is because $\sim$ does not simply switch sides for their notion of validity, as it does for ours.) So

INITIAL SEQUENTS FROM $\mathcal{L}$

As initial sequents, take all $[\Gamma \succ \Delta] \in \mathcal{A}(V)$ with $\Gamma \cup \Delta \subseteq \mathcal{L}$.

STRUCTURAL RULE

$$\text{D: } \frac{[\Gamma \succ \Delta]}{[\Gamma', \Gamma \succ \Delta, \Delta']}$$

TRUTH RULES

$$\bot\text{L: } \frac{}{[\bot \succ\ ]} \qquad \top\text{R: } \frac{}{[\ \succ \top]}$$

$$\sim\text{L: } \frac{[\Gamma \succ \Delta, \phi]}{[\sim\phi, \Gamma \succ \Delta]} \qquad \sim\text{R: } \frac{[\phi, \Gamma \succ \Delta]}{[\Gamma \succ \Delta, \sim\phi]}$$

$$\curlywedge\text{L: } \frac{[\phi, \psi, \Gamma \succ \Delta]}{[\phi \curlywedge \psi, \Gamma \succ \Delta]} \qquad \curlywedge\text{R: } \frac{[\Gamma \succ \Delta, \phi] \qquad [\Gamma \succ \Delta, \psi]}{[\Gamma \succ \Delta, \phi \curlywedge \psi]}$$

$$\curlyvee\text{L: } \frac{[\phi, \Gamma \succ \Delta] \qquad [\psi, \Gamma \succ \Delta]}{[\phi \curlyvee \psi, \Gamma \succ \Delta]} \qquad \curlyvee\text{R: } \frac{[\Gamma \succ \Delta, \phi, \psi]}{[\Gamma \succ \Delta, \phi \curlyvee \psi]}$$

INFORMATIONAL RULES

$$*\text{L: } \frac{}{[* \succ\ ]} \qquad *\text{R: } \frac{}{[\ \succ *]}$$

$$\sqcap\text{L: } \frac{[\phi, \psi, \Gamma \succ \Delta]}{[\phi \sqcap \psi, \Gamma \succ \Delta]} \qquad \sqcap\text{R: } \frac{[\Gamma \succ \Delta, \phi, \psi]}{[\Gamma \succ \Delta, \phi \sqcap \psi]}$$

$$\sqcup\text{L: } \frac{[\phi, \Gamma \succ \Delta] \qquad [\psi, \Gamma \succ \Delta]}{[\phi \sqcup \psi, \Gamma \succ \Delta]} \qquad \sqcup\text{R: } \frac{[\Gamma \succ \Delta, \phi] \qquad [\Gamma \succ \Delta, \psi]}{[\Gamma \succ \Delta, \phi \sqcup \psi]}$$

Figure 2. Sequents for $\mathcal{L}_{\sqsubseteq}$

Unfortunately, adding $\neg$ to this system is not simply a matter of adding a left rule and a right rule. Instead, we look one connective down, adding rules that govern the interaction of $\neg$ with each other connective. These rules are thus much like the rules given in [3, 41] for the logic FDE, or the rules discussed in footnote 26 for the truth vocabulary. (See also [2, p. 39] for rules for $\neg$ that take a similar strategy.) We also add new initial sequents, recording $\neg$'s behaviour as applied directly to formulas of $\mathcal{L}$.

Footnote 26 continued
while we have, for example, $\sqcap$ with the left rule of $\curlywedge$ and the right rule of $\curlyvee$, they have $\sqcap$ with the *unembedded* rules of $\curlywedge$ and the *under-*$\sim$ rules of $\curlyvee$.

INITIAL SEQUENTS FOR LITERALS

As initial sequents, take all $[\Gamma \succ \Delta] \in \mathcal{A}(V)$ with $\Gamma \cup \Delta \subseteq \mathcal{L} \cup \neg\mathcal{L}$.

$\neg$ RULES

$$\neg\bot\mathrm{L}: \frac{}{[\neg\bot \succ\ ]} \qquad \neg\top\mathrm{R}: \frac{}{[\ \succ \neg\top]} \qquad \neg\mathtt{I}\mathrm{L}: \frac{}{[\neg\mathtt{I} \succ\ ]} \qquad \neg\mathtt{I}\mathrm{R}: \frac{}{[\ \succ \neg\mathtt{I}]}$$

$$\neg{\sim}\mathrm{L}: \frac{[\Gamma \succ \Delta, \neg\phi]}{[\neg{\sim}\phi, \Gamma \succ \Delta]} \qquad \neg{\sim}\mathrm{R}: \frac{[\neg\phi, \Gamma \succ \Delta]}{[\Gamma \succ \Delta, \neg{\sim}\phi]}$$

$$\neg\wedge\mathrm{L}: \frac{[\neg\phi, \neg\psi, \Gamma \succ \Delta]}{[\neg(\phi \wedge \psi), \Gamma \succ \Delta]} \qquad \neg\wedge\mathrm{R}: \frac{[\Gamma \succ \Delta, \neg\phi] \qquad [\Gamma \succ \Delta, \neg\psi]}{[\Gamma \succ \Delta, \neg(\phi \wedge \psi)]}$$

$$\neg\vee\mathrm{L}: \frac{[\neg\phi, \Gamma \succ \Delta] \qquad [\neg\psi, \Gamma \succ \Delta]}{[\neg(\phi \vee \psi), \Gamma \succ \Delta]} \qquad \neg\vee\mathrm{R}: \frac{[\Gamma \succ \Delta, \neg\phi, \neg\psi]}{[\Gamma \succ \Delta, \neg(\phi \vee \psi)]}$$

$$\neg\sqcap\mathrm{L}: \frac{[\neg\phi, \Gamma \succ \Delta] \qquad [\neg\psi, \Gamma \succ \Delta]}{[\neg(\phi \sqcap \psi), \Gamma \succ \Delta]} \qquad \neg\sqcap\mathrm{R}: \frac{[\Gamma \succ \Delta, \neg\phi] \qquad [\Gamma \succ \Delta, \neg\psi]}{[\Gamma \succ \Delta, \neg(\phi \sqcap \psi)]}$$

$$\neg\sqcup\mathrm{L}: \frac{[\neg\phi, \neg\psi, \Gamma \succ \Delta]}{[\neg(\phi \sqcup \psi), \Gamma \succ \Delta]} \qquad \neg\sqcup\mathrm{R}: \frac{[\Gamma \succ \Delta, \neg\phi, \neg\psi]}{[\Gamma \succ \Delta, \neg(\phi \sqcup \psi)]}$$

$$\neg\neg\mathrm{L}: \frac{[\phi, \Gamma \succ \Delta]}{[\neg\neg\phi, \Gamma \succ \Delta]} \qquad \neg\neg\mathrm{R}: \frac{[\Gamma \succ \Delta, \phi]}{[\Gamma \succ \Delta, \neg\neg\phi]}$$

Figure 3. Extra rules for $\neg$

The needed rules are in Figure 3; combining these rules with those from Figure 2 gives our calculus for $\mathcal{L}_+$. Comparing the rules in Figure 2 for each monotonic connective on its own with the rules for its interaction with $\neg$ in Figure 3 reveals a pattern: $\neg$ is in a sense transparent to the truth connectives, which continue to obey essentially the same rules as in unembedded occurrences, simply passing the $\neg$ up from their subformulas; and $\neg$ reverses the information connectives, trading $\sqcap$ for $\sqcup$ and $\mathtt{I}$ for $*$, and vice versa.

3.3. Soundness and Partial Completeness

Here, we discuss the full proof system for $\mathcal{L}_+$, involving all initial sequents and rules from both Figures 2 and 3. (We will return to the system of Figure 2 alone for $\mathcal{L}_\sqsubseteq$ presently.)

THEOREM 6. *Given $V \subseteq \mathfrak{V}_4$, if $[\Gamma \succ \Delta]$ is derivable in the above calculus (with initial sequents sensitive to V), then $[\Gamma \succ \Delta] \in \mathcal{A}_{\mathrm{SS}}(V)$.*

PROOF. This is as usual for a soundness proof: the initial sequents are all in $\mathcal{A}_{\text{SS}}(V)$, and each rule preserves this property. We can be confident that each rule preserves this property regardless of the choice of V because each rule is *locally* sound: any valuation that is a counterexample to the conclusion-sequent of an application any rule is *itself* a counterexample to some premise-sequent of that application of the rule. ■

Unfortunately, as we will show in a moment, this system is not complete. However, it is complete for an important class of arguments: the connective-finite arguments.

DEFINITION 6. An argument $[\Gamma \succ \Delta]$ is *connective-finite* iff $(\Gamma \cup \Delta) \setminus (\mathcal{L} \cup \neg\mathcal{L})$ is finite.

The connective-finite arguments are those that contain only a finite number of connectives, as a connective-finite argument has only finitely many formulas containing connectives (with $\neg$ applied directly to members of $\mathcal{L}$ not counting), each of which contain only finitely many connectives. It is also worth noting that the sequent system in question can *only* derive connective-finite sequents: all initial sequents are connective-finite, and all rules preserve connective-finitude.

THEOREM 7. *This sequent system is complete for connective-finite arguments: every connective-finite* $[\Gamma \succ \Delta] \in \mathcal{A}_{\text{SS}}(V)$ *has a derivation in this calculus (with initial sequents sensitive to* V*).*

PROOF. Build a reduction tree in the usual way from any such sequent $[\Gamma_0 \succ \Delta_0]$, by running the rules of the proof system backwards, and closing any branch if it reaches a sequent that can be derived from an initial sequent by an application of the dilution rule D. If it closes, the result is (near enough, modulo the just-mentioned applications of dilution) a derivation of $[\Gamma_0 \succ \Delta_0]$. So suppose it has an open completed branch. Take the leaf sequent of the branch; let this be $[\Gamma \succ \Delta]$. We give a valuation $v \in V$ with $v \divideontimes [\Gamma \succ \Delta]$.

For any $\phi \in \mathcal{L}$, set $v(\phi)$ as follows:

- if $\phi \in \Gamma$, then $v(\phi) \in \{\top, \mathrm{I}\}$,
- if $\phi \in \Delta$, then $v(\phi) \in \{\bot, \mathrm{I}\}$,
- if $\neg\phi \in \Gamma$, then $v(\phi) \in \{\top, *\}$,
- if $\neg\phi \in \Delta$, then $v(\phi) \in \{\bot, *\}$.

Because of our initial sequents for literals, we can be sure there is such a $v \in V$; if there were not, the branch would have closed.[27]

We want to show that the clauses specified above for $\phi \in \mathcal{L}$ extend to the full language; they hold for all $\phi \in \mathcal{L}_+$. If this is so, we have (by the first two clauses) $v \divideontimes [\Gamma \succ \Delta]$; since $[\Gamma_0 \succ \Delta_0] \sqsubseteq [\Gamma \succ \Delta]$, it follows that $v \divideontimes [\Gamma_0 \succ \Delta_0]$, and we're done.

Base case: The atoms are all set, since they were taken care of in the specification of v.

Constants: Informational: we need to be sure that neither $*$ nor $\neg\text{I}$ appears in $\Gamma \cup \Delta$, since this would violate our clauses. Initial sequents among the informational rules take care of this. Truth: we need to be sure that neither $\bot$ nor $\neg\bot$ is in Γ, and that neither $\top$ nor $\neg\top$ is in Δ. Initial sequents among the truth and interaction rules take care of this.

Binary connectives: For $\curlywedge$: if $\phi \curlywedge \psi \in \Gamma$, then $\phi, \psi \in \Gamma$ by construction. By the inductive hypothesis, $v(\phi), v(\psi) \in \{\top, \text{I}\}$; and so $v(\phi \curlywedge \psi) \in \{\top, \text{I}\}$ as well. If $\phi \curlywedge \psi \in \Delta$, then either $\phi \in \Delta$ or $\psi \in \Delta$ by construction. By the inductive hypothesis, either $v(\phi) \in \{\bot, \text{I}\}$ or $v(\psi) \in \{\bot, \text{I}\}$. Either way, $v(\phi \curlywedge \psi) \in \{\bot, \text{I}\}$. If $\neg(\phi \curlywedge \psi) \in \Gamma$, then $\neg\phi, \neg\psi \in \Gamma$, and the rest is similar to the $\phi \curlywedge \psi$ case. If $\neg(\phi \curlywedge \psi) \in \Delta$, then either $\neg\phi \in \Delta$ or $\neg\psi \in \Delta$, and the rest is similar to the $\phi \curlywedge \psi$ case.

For $\curlyvee$: just as for $\curlywedge$.

For $\sqcap$: if $\phi \sqcap \psi \in \Gamma$, then $\phi, \psi \in \Gamma$ by construction. By the induction hypothesis, $v(\phi), v(\psi) \in \{\top, \text{I}\}$; hence $v(\phi \sqcap \psi) \in \{\top, \text{I}\}$. The case where $\phi \sqcap \psi \in \Delta$ is similar. Now, if $\neg(\phi \sqcap \psi) \in \Gamma$, then either $\neg\phi \in \Gamma$ or $\neg\psi \in \Gamma$ by construction. By the induction hypothesis, either $v(\phi) \in \{\top, *\}$ or $v(\psi) \in \{\top, *\}$. Either way, $v(\phi \sqcap \psi) \in \{\top, *\}$, and hence $v(\neg(\phi \sqcap \psi)) \in \{\top, *\}$. The case where $\neg(\phi \sqcap \psi) \in \Delta$ is similar.

For $\sqcup$: just as for $\sqcap$.

Unary connectives: For $\sim$: if $\sim\phi \in \Gamma$, then $\phi \in \Delta$ by construction. By the inductive hypothesis, $v(\phi) \in \{\bot, \text{I}\}$, and so $v(\sim\phi) \in \{\top, \text{I}\}$. If $\sim\phi \in \Delta$, then $\phi \in \Gamma$ by construction. By the inductive hypothesis, $v(\phi) \in \{\top, \text{I}\}$, and so $v(\sim\phi) \in \{\bot, \text{I}\}$. If $\neg\sim\phi \in \Gamma$, then $\neg\phi \in \Delta$ by construction, and the rest is like the $\sim\phi$ case. If $\neg\sim\phi \in \Delta$, then $\neg\phi \in \Gamma$ by construction, and the rest is like the $\sim\phi$ case.

[27] This is where the restriction to connective-finite sequents is needed. Without such a restriction, we would potentially have an infinite open branch; it would then have no leaf sequent. The natural move would be to take the union of its sequents as $[\Gamma \succ \Delta]$. But even if each sequent in the branch has a counterexample in V, it does not follow in general that their union does as well. So long as there are only finitely many sentences in $(\Gamma \cup \Delta) \setminus (\mathcal{L} \cup \neg\mathcal{L})$, however, each branch will have a finite length whether open or closed.

For $\neg$: if $\neg\phi \in \Gamma$, then by the induction hypothesis $v(\phi) \in \{\top, *\}$, and so $v(\neg\phi) \in \{\top, \bot\}$. The case where $\neg\phi \in \Delta$ is similar. Now, if $\neg\neg\phi \in \Gamma$, then by construction $\phi \in \Gamma$. By the induction hypothesis, $v(\phi) \in \{\top, \bot\}$, and so $v(\neg\phi) \in \{\top, *\}$. The case where $\neg\neg\phi \in \Delta$ is similar. ■

This does not give us a full description of $\mathcal{A}_{\text{SS}}(V)$ for $V \subseteq \mathfrak{V}_4$, but it goes a long way towards such a thing: we have a sound and complete calculus for connective-finite arguments, and a sound calculus for arguments in general.

Unfortunately, it is not simply that the above completeness proof does not work for other sequents: indeed the system is not complete in general. To see this, let $\mathcal{L}$ be the natural numbers $\mathbb{N}$, let v_n be the valuation that assigns $\top$ to each natural number $\leq n$ and $\bot$ to the rest, and let $V = \{v_n \mid n \in \mathbb{N}\}$. Then for each n, $[0, \ldots, n \succ] \notin \mathcal{A}_{\text{SS}}(V)$, but $[\mathbb{N} \succ] \in \mathcal{A}_{\text{SS}}(V)$. Now, consider the argument $[1 \curlywedge 2, 3 \curlywedge 4, 5 \curlywedge 6, \ldots \succ]$. This too is in $\mathcal{A}_{\text{SS}}(V)$; it has the same counterexamples as $[\mathbb{N} \succ]$. But there is no derivation of it in our system; any such derivation would have to apply the $\curlywedge$L rule infinitely many times, which is not possible in a finite derivation. So the system as it stands is indeed not complete.

Although we do not have completeness in general, we can use Theorems 6 and 7 to achieve a Cut-admissibility result, at least for an appropriate form of Cut. Usual forms of Cut are out of the question, as they encode the very kind of transitivity that the value $*$ allows us to evade. But consider the following form of the Cut rule:

$$\text{Cut:} \quad \frac{[\Gamma \succ \Delta, \phi] \qquad [\phi, \Gamma \succ \Delta] \qquad [\neg\phi, \Gamma \succ \Delta, \neg\phi]}{[\Gamma \succ \Delta]}$$

THEOREM 8. *The rule Cut is admissible in the above sequent system (with initial sequents sensitive to V).*

PROOF. Suppose there are derivations in the system of the three premise sequents of an application of Cut. Then these premise sequents must be connective-finite, since every sequent derivable in this system is connective-finite; and they must be in $\mathcal{A}_{\text{SS}}(V)$, by Theorem 6. As can be checked, the rule of Cut is sound for any such V: whenever its premise sequents are in $\mathcal{A}_{\text{SS}}(V)$, so too is its conclusion sequent. So $[\Gamma \succ \Delta] \in \mathcal{A}_{\text{SS}}(V)$. And since the premise-sequents are connective-finite, so too must $[\Gamma \succ \Delta]$ be; thus, it has a derivation, by Theorem 7. ■

3.4. Restricted Languages

We can use this calculus to give similar results for languages that don't contain the full stock of connectives we've considered.

Note first that while the full system lacks the subformula property, it has a weakening of this property, which we might call the *connective-occurrence property*: if a connective appears in a sequent in a derivation, then it also appears in the endsequent of the derivation.[28] This can be determined by inspecting the rules: the conclusion-sequent of any occurrence of any rule must contain all the connectives contained in any premise-sequents of the rule.

This is enough to allow us to prune our proof system down to handle languages without the full stock of connectives we have considered. For any subset $\boldsymbol{S}$ of our full stock of connectives, let $\mathcal{L}_{\boldsymbol{S}}$ be the language determined by closing $\mathcal{L}$ under the connectives in $\boldsymbol{S}$. Say that a connective *figures* in a rule iff it occurs in the conclusion-sequent of every application of the rule.[29]

Now, let the calculus determined by $\boldsymbol{S}$ (the $\boldsymbol{S}$*-restricted calculus*) be a restriction of the full calculus determined as follows: remove all rules in which figure any connective not in $\boldsymbol{S}$, and remove any initial sequents containing any formula not in $\mathcal{L}_{\boldsymbol{S}}$.[30]

THEOREM 9. *For any $V \subseteq \mathfrak{V}_4$ and any subset $\boldsymbol{S}$ of our full stock of connectives, the $\boldsymbol{S}$-restricted calculus is sound for $\mathcal{A}_{\text{SS}}(V)$ in $\mathcal{L}_{\boldsymbol{S}}$, and complete for connective-finite arguments in $\mathcal{A}_{\text{SS}}(V)$ in $\mathcal{L}_{\boldsymbol{S}}$.*

PROOF. For soundness: take any derivation in the restricted system. Since the full system is sound, and the full system contains this derivation, the derived argument is in $\mathcal{A}_{\text{SS}}(V)$.

For completeness: suppose some connective-finite $[\Gamma \succ \Delta]$ in $\mathcal{L}_{\boldsymbol{S}}$ is in $\mathcal{A}_{\text{SS}}(V)$. By completeness for the full system, there is some derivation in the full system of $[\Gamma \succ \Delta]$. By the connective-occurrence property, every connective occurring in the derivation must be in $\boldsymbol{S}$. But then the derivation cannot contain any application of a rule in which figures any connective not in $\boldsymbol{S}$, nor can it contain any initial sequents not drawn from $\mathcal{L}_{\boldsymbol{S}}$. So the derivation in question is in fact a derivation in the $\boldsymbol{S}$-restricted calculus. ∎

[28]Indeed, the system has two other weakenings of the subformula property that can prove useful in analyzing proofs, and which make clear some of the ways in which it is $\neg$ responsible for breaking the ordinary subformula property. First, any formula appearing in a derivation of $[\Gamma \succ \Delta]$ is either a subformula of some formula in $\Gamma \cup \Delta$, or the information negation of some such subformula. Second, if $\neg$ does not occur in $[\Gamma \succ \Delta]$, then any formula appearing in a derivation of $[\Gamma \succ \Delta]$ is a subformula of some formula in $\Gamma \cup \Delta$.

[29]In this setting, this is equivalent to: it appears in the schematic conclusion-sequent we have given. For that matter, it is also (here) equivalent to: it appears in the rule's name.

[30]So if $\neg \in \boldsymbol{S}$, then we have all initial sequents from the full calculus, and if $\neg \notin \boldsymbol{S}$, we have only those initial sequents drawn fully from $\mathcal{L}$.

As a special case, Theorem 9 gives us soundness and connective-finite completeness for the system of Figure 2 and the language $\mathcal{L}_{\sqsubseteq}$. This system does have the full subformula property. It really is, then, only $\neg$ among the connectives of $\mathcal{L}_{+}$ causing the complications; this gives some further reason to be suspicious of non-information-monotonic connectives. (However, without $\neg$ the rule Cut, in the form we've shown admissible, cannot be stated.)

4. Conclusion

Attending to the Galois connections between sets of arguments and sets of valuations has proved useful in a range of applications to logics that are reflexive, monotonic, and completely transitive, which can all be handled through the lens of bivaluations. In this paper, we've shown how to extend this toolkit to logics that might fail to be reflexive or completely transitive or both, by adding up to two more values to the valuations. Finally, we considered the behaviour of connectives witnessing the bilattice structure of the resulting tetravaluations.

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