Vagueness and nonclassical probability

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Vagueness

The AAP is a longstanding organisation:
founded in 1922.

But longstandingness is not a property that draws a clean line:

when an organisation is not yet longstanding,
waiting an hour more can't make it longstanding.

So since the AAP is longstanding now,
it must have been longstanding an hour ago,
and an hour before that, and an hour before that…

This is the sorites paradox,
and it too is longstanding.

Most of our categories are sensitive to large differences without being sensitive to small differences.

But a large difference is just a collection of small differences.

It’s easy to ignore paradoxes like ‘this sentence is false’;
they don’t come up most of the time.

The sorites is not like that:
we grapple with vague distinctions all the time.

And we do it fluently!

These issues could arise in almost everything we say, and for the most part they just don’t.

How do we do it?

A subsidiary goal:

to show how logic, probability, and experimentation can inform each other.

Sharp lines

Some deny the appearances
and just maintain that every category does draw sharp lines.

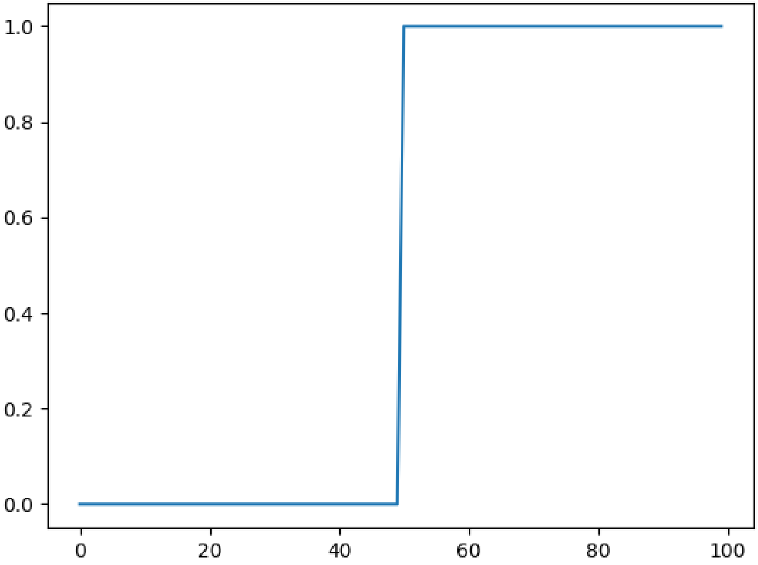
On this view, there was an instant at which the AAP
suddenly leapt into longstandingness all at once.

This is the Matthew 5:37 theory of vagueness:

Matthew 5:37 (NKJV)

"But let your 'Yes' be 'Yes,' and your 'No,' 'No.' For whatever is more than these is from the evil one."

Or, as bible.art puts it: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This is not intended to represent a chosen policy
about using the word ‘longstanding’.

It is meant to be an already-existing fact
about the distinction the word draws.

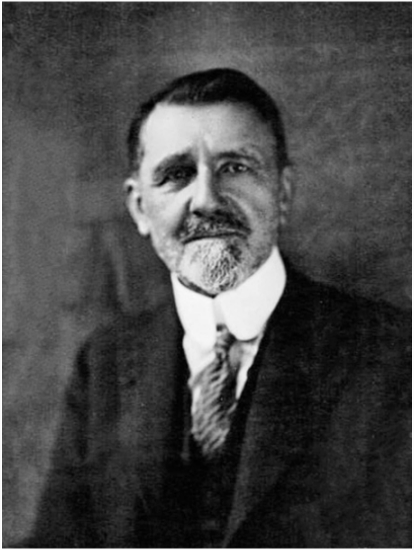
Exactly which instant was the key one?
Clearly we don’t know.

And so this view often comes with a package of ideas around vagueness and knowledge.

One idea is that there is only one real sharp line
between longstandingness and its absence,

but there are many sharp lines
that might, for all we know, be that real one.

Degrees of belief



Our uncertainty about where the sharp line lies comes in degrees.

I’m pretty sure the sharp line isn’t in 1800 or 1925, and that it isn’t in 2021 or 2050.

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I’m pretty sure the sharp line isn’t in 1800 or 1925, and that it isn’t in 2021 or 2050.

I don’t think it was in 1950 either, but I’m less sure. And 1973? Hell if I know.

It's customary to represent degrees of belief as numbers between 0 and 1, inclusive.

To believe something to degree:

1	is to be absolutely certain of it
0	is to be absolutely certain against it
.5	is to be completely undecided about it
.75	is to be halfway between undecided & certain, etc

A handy way to think about this uses valuations from classical logic, using the values 1 and 0.

Take a flip of a fair coin.
Let h be the claim the coin comes up heads,
and let t be the claim that it comes up tails.

Then we have four valuations, and we can do a truth table:

h	t	$h \vee t$	$h \wedge t$	
1	1	1	1	
1	0	1	0	
0	1	1	0	
0	0	0	0	

Then we have four valuations, and we can do a truth table:

h	t	$h \vee t$	$h \wedge t$	P
1	1	1	1	0
1	0	1	0	.499
0	1	1	0	.499
0	0	0	0	.002

(totalling 1)

h	t	$h \vee t$	$h \wedge t$	$\textsc{p}$
1	1	1	1	0
1	0	1	0	.499
0	1	1	0	.499
0	0	0	0	.002

What is $\textsc{p}(h \vee t)$?

It's $1 \times 0 + 1 \times .499 + 1 \times .499 + 0 \times .002$

h	t	$h \vee t$	$h \wedge t$	P
1	1	1	1	0
1	0	1	0	.499
0	1	1	0	.499
0	0	0	0	.002

What is $\text{P}(h \vee t)$?

It's $1 \times 0 + 1 \times .499 + 1 \times .499 + 0 \times .002$

h	t	$h \vee t$	$h \wedge t$	P
1	1	1	1	0
1	0	1	0	.499
0	1	1	0	.499
0	0	0	0	.002

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It's $1 \times 0 + 1 \times .499 + 1 \times .499 + 0 \times .002$

h	t	$h \vee t$	$h \wedge t$	P
1	1	1	1	0
1	0	1	0	.499
0	1	1	0	.499
0	0	0	0	.002

What is $\mathrm{P}(h \vee t)$?

It's $1 \times 0 + 1 \times .499 + 1 \times .499 + 0 \times .002$

h	t	$h \vee t$	$h \wedge t$	$\textsc{p}$
1	1	1	1	0
1	0	1	0	.499
0	1	1	0	.499
0	0	0	0	.002

What is $\textsc{p}(h \vee t)$?

It's $0 + .499 + .499 + 0 = .998$

Take someone's belief state to be represented by this kind of weighted truth table.

(JARGON: it is a convex sum of valuations)

Then we can read off their degrees of belief in complex statements like this.

For finite languages, there's a nice result:
Someone's belief state B is a convex sum of classical valuations iff:

$$\mathrm{B}(p \vee \neg p) = 1 \text{ and } \mathrm{B}(q \wedge \neg q) = 0$$

$$\text{If } A \vDash_{CL} B \text{ then } \mathrm{B}(A) \leq \mathrm{B}(B)$$

$$\mathrm{B}(A \vee B) + \mathrm{B}(A \wedge B) = \mathrm{B}(A) + \mathrm{B}(B)$$

Probabilities and vagueness

Back to vagueness.

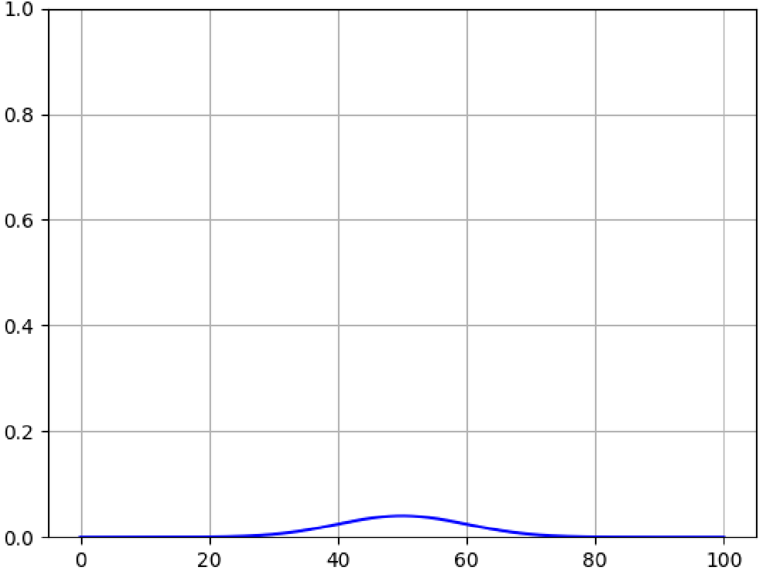
For any number i, let p_i be the claim
that the AAP was longstanding at the start of its ith hour of existence

There's been over 800,000 hours so far, so I'll skip the truth table!
(it would have over $2^{800,000}$ lines)

A simplifying assumption:
any valuation that has the AAP not being longstanding
at some hour after an hour where it is longstanding
is ruled out (gets weight 0).

All the valuations we care about pick some hour h,
assign 0 to p_i for all $i < h$,
and assign 1 to p_i for all $i \geq h$

This is still too many to show individually (over 800,000).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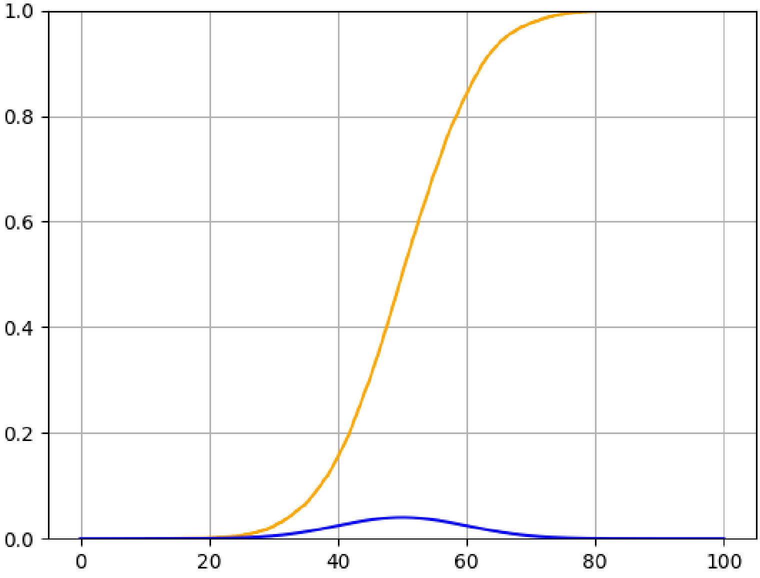
Suppose someone's belief state was like this.

What would they think of the various claims p_i?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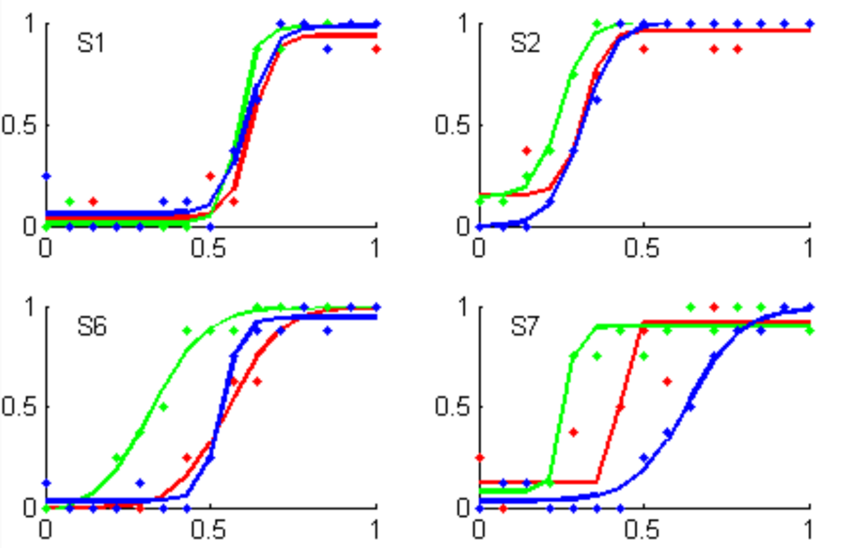
[Égré et al., 2013]

Now, here are some actual experimental results. Not bad!



This is all built on an underlying classical, sharp-lines base.

This speaker behaviour does not draw sharp lines,
but it looks similar to what we'd expect from probabilities
of sharp lines.

There is still an issue of interpretation:
on this approach, if we knew which valuation was the right one,
we should have only degrees of belief 1 and 0.

Fully informed would mean fully opinionated.

This strikes some (like me) as really implausible,
and nothing yet has eased that.

But perhaps it can be eased.

(Borel, 1907):

"I would like to try and show that one can give these questions [related to the sorites] a perfectly clear answer, provided one makes systematic use of the principles of probability. The questions are badly put, if one requires a yes or no answer; the true answer is a coefficient of probability" [see [Égré and Barberousse, 2014]]

But perhaps it can be eased.

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[Edgington, 1992]:

"I do not say that the indeterminacy of vague concepts is an epistemic matter. There exist different applications of probabilistic structure. Objective chances apply if and when the future is physically undetermined by the past. Relative frequencies also satisfy the principles of probability. I propose that it is also the right structure for theorizing about the indeterminacy of application of vague concepts."

So perhaps there are more plausible interpretations than the epistemic one available.

Once we take degrees of belief into account, a binary sharp-lines model starts to look okay as a model of some of the evidence we have.

Nonclassicality

The story doesn’t end there, though.

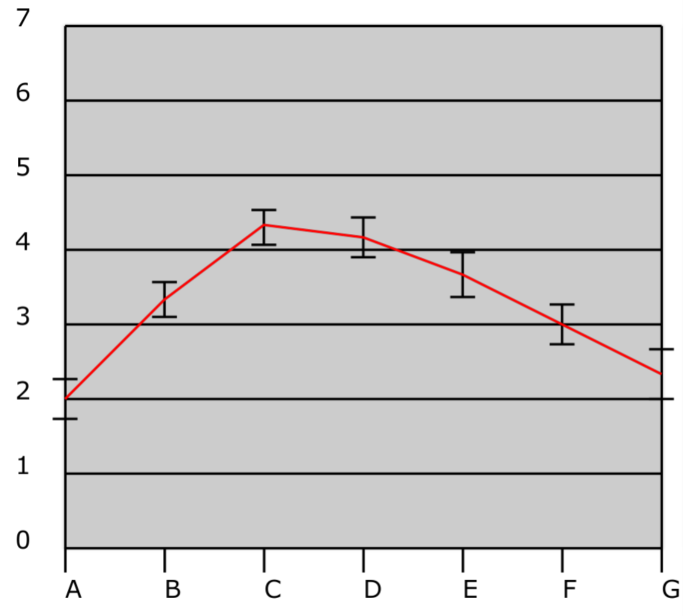
There’s other evidence that fits less well
with this binary picture,
even in its probabilised version.

This has to do with borderline contradictions:
sentences that say that something both is and isn't a certain way,
when it is a borderline case of being that way.

If someone says that a colour both is and isn't red,
they likely mean to communicate that it's a borderline case of red.

This, too, is reflected in experimental results.

[Ripley, 2011]



[Alxatib and Pelletier, 2011]

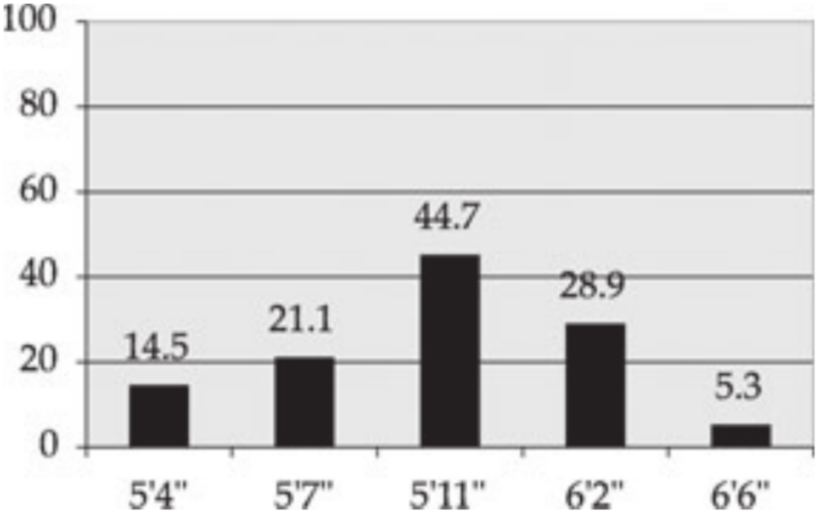
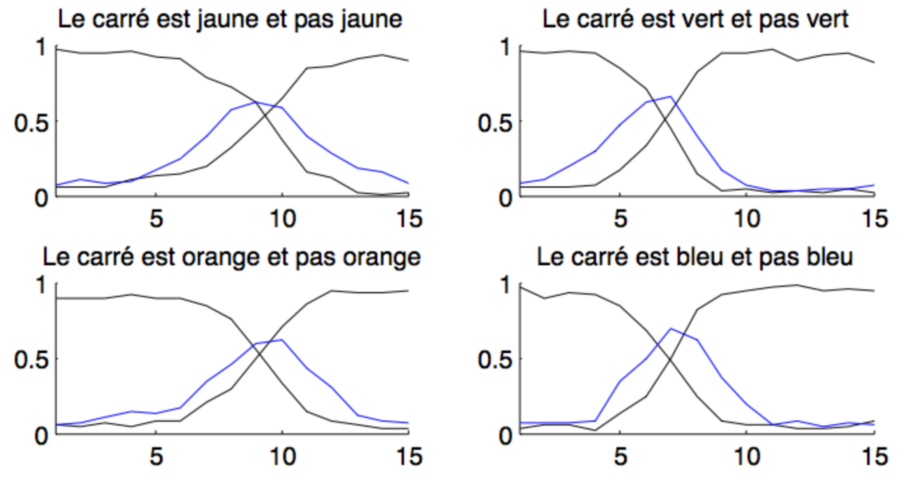


Figure 8 *'True' responses to 'x is tall and not tall'*

[Égré et al., 2013]



But $p \wedge \neg p$ always has probability 0
on a classical way of handling probability.

A simple classical approach, even when probabilised,
predicts flat disagreement with all such claims.

Many-valued probability

Maybe 1 and 0 aren't enough;
maybe we've missed some truth values.

But how many truth values are there?

I'll take up Biden's answer:
there are at least three.

I'll use 1, .5, 0;
you might also think of $[0, 1]$

I'll use a common scheme for valuations for compound sentences:

$$v(\neg A) = 1 - v(A)$$
$$v(A \wedge B) = \text{the minimum of } v(A), v(B)$$
$$v(A \vee B) = \text{the maximum of } v(A), v(B)$$

(our classical valuations are special cases)

You know that Alice and Zebra's heights total 330cm,
with no further information.

It could be that one is clearly tall (say 170) and one borderline (160),
or that both are borderline (say 165),
or that one is clearly not tall (say 140) and one clearly tall (190).

Let's say those are the possibilities.

Let a be the claim that Alice is tall and z the claim that Zebra is.

a	z	$a \wedge z$	P
1	1	1	0
1	.5	.5	.2
1	0	0	.2
.5	1	.5	.2
.5	.5	.5	.2
.5	0	0	0
0	1	0	.2
0	.5	0	0
0	0	0	0

What is your degree of belief in $a \wedge z$?

Let a be the claim that Alice is tall and z the claim that Zebra is.

a	z	$a \wedge z$	P
1	1	1	0
1	.5	.5	.2
1	0	0	.2
.5	1	.5	.2
.5	.5	.5	.2
.5	0	0	0
0	1	0	.2
0	.5	0	0
0	0	0	0

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.5	.5	.5	.2
.5	0	0	0
0	1	0	.2
0	.5	0	0
0	0	0	0

What is your degree of belief in $a \wedge z$?

$.5 \times .2$

Let a be the claim that Alice is tall and z the claim that Zebra is.

a	z	$a \wedge z$	P
1	1	1	0
1	.5	.5	.2
1	0	0	.2
.5	1	.5	.2
.5	.5	.5	.2
.5	0	0	0
0	1	0	.2
0	.5	0	0
0	0	0	0

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.5	.5	.5	.2
.5	0	0	0
0	1	0	.2
0	.5	0	0
0	0	0	0

What is your degree of belief in $a \wedge z$?

$.5 \times .2 + .5 \times .2$

Let a be the claim that Alice is tall and z the claim that Zebra is.

a	z	$a \wedge z$	P
1	1	1	0
1	.5	.5	.2
1	0	0	.2
.5	1	.5	.2
.5	.5	.5	.2
.5	0	0	0
0	1	0	.2
0	.5	0	0
0	0	0	0

What is your degree of belief in $a \wedge z$?

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1	0	0	.2
.5	1	.5	.2
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0	1	0	.2
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0	0	0	0

What is your degree of belief in $a \wedge z$?

$$.5 \times .2 + .5 \times .2 + .5 \times .2$$

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1	0	0	.2
.5	1	.5	.2
.5	.5	.5	.2
.5	0	0	0
0	1	0	.2
0	.5	0	0
0	0	0	0

What is your degree of belief in $a \wedge z$?

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.5	1	.5	.2
.5	.5	.5	.2
.5	0	0	0
0	1	0	.2
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0	0	0	0

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.5	.5	.5	.2
.5	0	0	0
0	1	0	.2
0	.5	0	0
0	0	0	0

What is your degree of belief in $a \wedge z$?

$$.5 \times .2 + .5 \times .2 + .5 \times .2$$

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1	0	0	.2
.5	1	.5	.2
.5	.5	.5	.2
.5	0	0	0
0	1	0	.2
0	.5	0	0
0	0	0	0

What is your degree of belief in $a \wedge z$?

$$.5 \times .2 + .5 \times .2 + .5 \times .2 = .3$$

(Gil Sanchez et al; 2023, 2024)

$\textsc{b}$ is a convex sum of three-valued valuations iff:

$$\text{if } A \vDash_{S3} B, \text{ then } \textsc{b}(A) \leq \textsc{b}(B)$$

$$\textsc{b}(\neg A) = 1 - \textsc{b}(A)$$

$$\textsc{b}(A \vee B) + \textsc{b}(A \wedge B) = \textsc{b}(A) + \textsc{b}(B)$$

$$\textsc{b}(A) = \textsc{b}(A \wedge B) + \textsc{b}(A \wedge \neg B) - \textsc{b}(A \wedge \neg A \wedge B \wedge \neg B)$$

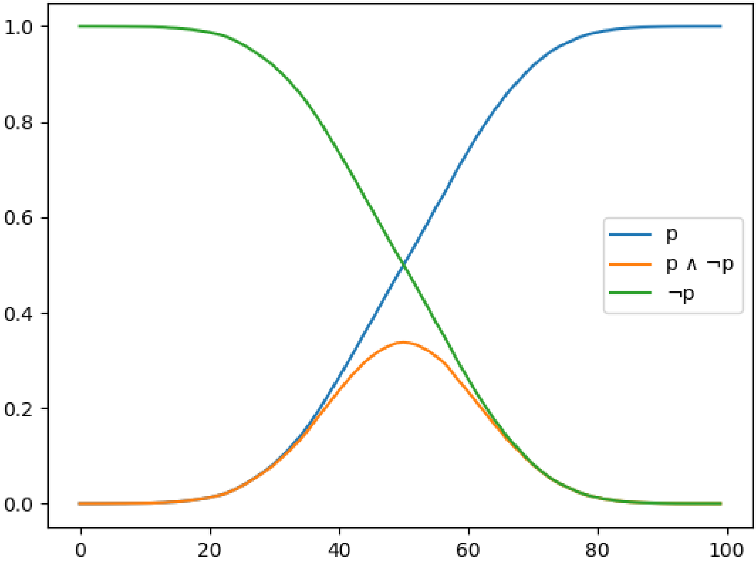
Pleasantly, this can be extended:
valuations into $[0,1]$ are themselves such $\textsc{b}$,
so can be arrived at as convex sums of three-valued valuations.

That means that any $\textsc{b}$ that is a convex sum of $[0,1]$ valuations
is already a convex sum of three-valued valuations.

This is roughly the approach to vague probability
recommended in [Smith, 2008].

Assume the centre of the borderline is distributed just like the threshold was before.

And assume that the width of the borderline is independently normally distributed as well.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p ∧ ¬p
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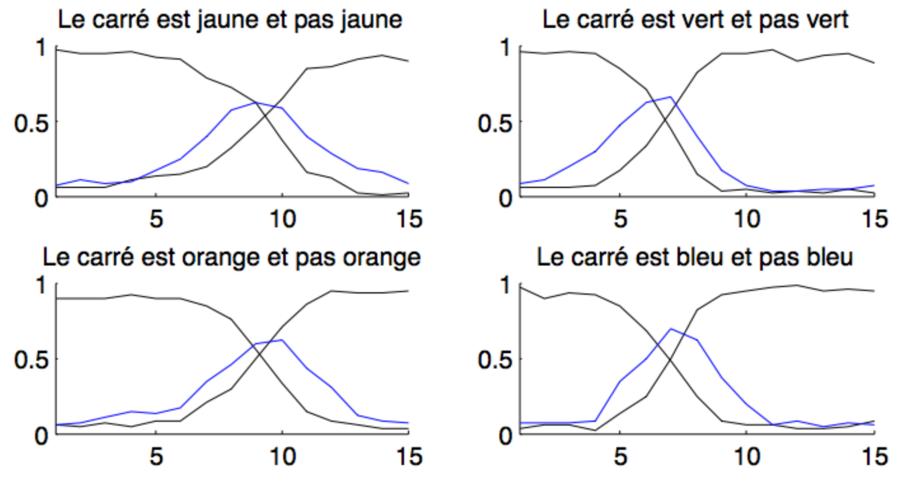
A final tweak

[Égré et al., 2013]



Judgments of $p \wedge \neg p$ often exceed
justgments of p or judgments of $\neg p$,
and sometimes both.

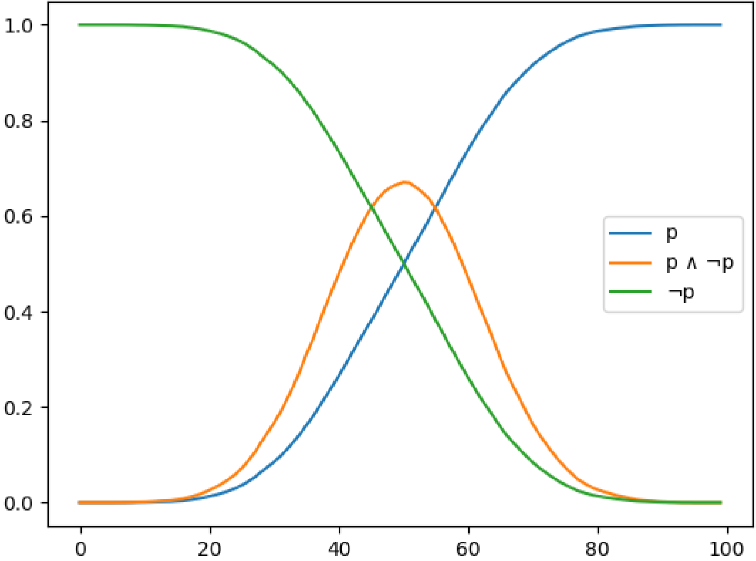
The setup so far cannot directly handle this,
since $p \wedge \neg p \vDash_{S3} p$ and $p \wedge \neg p \vDash_{S3} \neg p$

Whatever explains this, a plausible thought
is that it has something to do with the fact that $p \wedge \neg p$ can never take a truth value above .5, while p and $\neg p$ can.

So when $p \wedge \neg p$ does take the value .5, it's as good as it gets; not so for p and $\neg p$, which can make it all the way to 1.

A final thought: suppose the effect is pragmatic. Say credences are as above, but assertion works like this:

associate with each sentence A the maximum value $\textsc{m}(A)$ it can take, and let the probability of asserting A be $\textsc{b}(A)/\textsc{m}(A)$.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p
p ∧ ¬p
¬p

Conclusions

A sharp binary theory of vagueness can be made ok by considering probabilities.

Just ok and no better, though: it remains helpless to deal with borderline contradictions.

The same kind of probabilising works better with more truth values in the mix.

Still some trouble with the Alxatib & Pelletier effect, but a simple pragmatic overlay seems to help.

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