Vagueness and bilateral probabilistic consequence

(joint w Paul Egré)

Ellie Ripley (they/she)

Monash University
https://negation.rocks

The plan

Vagueness

Symmetric consequence

Similarity

Vagueness

Gradable adjectives like ‘tall’ are often given threshold semantics:

The comparative ‘taller’ is taken as more fundamental;
to be tall is then to be taller than some threshold.

Locating such a threshold is notoriously difficult.

Some options:

- there's a single fixed threshold
  (we just don't know where it is)
- there's a single threshold in each context
  (but it moves)
- there's a set of relevant thresholds
  (fixed or moving)

Our approach for today:

there is a probability distribution over thresholds.

(All probabilities today are classical.)

This can be interpreted epistemically if you like,
as uncertainty about where the single true threshold lies.

We don’t love that,
but we can’t stop you.

It's not the only option, though:

(Borel, 1907):

"I would like to try and show that one can give these questions [related to the sorites] a perfectly clear answer, provided one makes systematic use of the principles of probability. The questions are badly put, if one requires a yes or no answer; the true answer is a coefficient of probability"

It’s not the only option, though:

(Edgington, 1992)

“I do not say that the indeterminacy of vague concepts is an epistemic matter. There exist different applications of probabilistic structure. Objective chances apply if and when the future is physically undetermined by the past. Relative frequencies also satisfy the principles of probability. I propose that it is also the right structure for theorizing about the indeterminacy of application of vague concepts.”

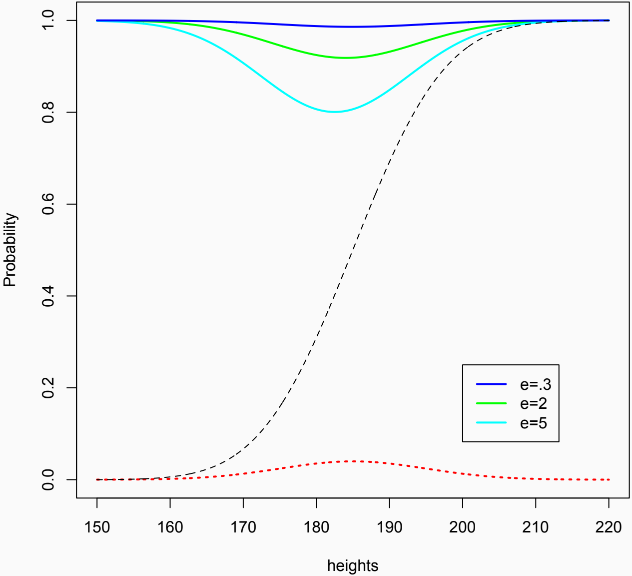
Take an n-step sorites sequence, with members $0 \ldots n$.

P(the threshold falls somewhere in the sequence) = 1;
P(the threshold is between i and $i + 1$) can be very low for all i

How low they can go depends on n;
finite additivity requires that the step-by-step probabilities sum to 1.

The probability of the material conditional
'if i is tall, then $i+1$ is tall'
is $1 - \mathrm{P}(\text{the threshold is between } i \text{ and } i+1)$

So all these tolerance conditionals
can have very high probability.



Red: threshold; Black: $P(Fx)$; Other: $P(F(x+e) \supset F(x))$

Symmetric consequence

A guiding idea:
we reason with less than certain principles
(like tolerance conditionals)

This reasoning should approach classical logic
as we require more and more certainty.

Let a high set α be such that:
$1 \in \alpha \subseteq (0, 1]$, and
α is an upset.

A high set represents a choice of probabilities that are high enough to reason from.

Fix a high set α, and think about arguments that preserve probability-in-α over all probability functions.

That is, count $[\Gamma \succ \Delta]$ as valid iff: there is no P with $P(\gamma) \in \alpha$ for all $\gamma \in \Gamma$ and $P(\delta) \notin \alpha$ for all $\delta \in \Delta$

Two problems:

- even at $\alpha = \{1\}$, we don't get Set-Set classical logic (just supervaluationist logic)
- no good sense in which even supervaluationist logic is a limit:
    - if α and β determine distinct consequence relations, they determine incomparable ones
    - most of your favourite arguments are invalid for all high sets other than $\{1\}$ (eg anything with two needed premises and a conclusion)

So preservation is not going to capture what we're after; forget it.

A different idea:

Given a high set α, let its corresponding low set be $\overline{\alpha} = \{x | 1 - x \in \alpha\}$

Where probabilities in α are high enough to reason from,
probabilities in $\overline{\alpha}$ are too low to reason to.

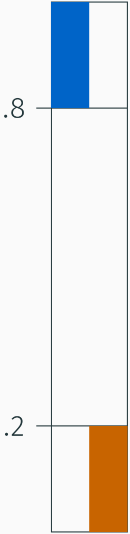
 Symmetric consequence

Now, count $[\Gamma \succ \Delta]$ as valid iff: there is no P
with $P(\gamma) \in \alpha$ for all $\gamma \in \Gamma$
and $P(\delta) \in \overline{\alpha}$ for all $\delta \in \Delta$

This is α-symmetric consequence, written $\Gamma \vDash^{\alpha} \Delta$

When $\Gamma \vDash^{\alpha} \Delta$,
as long as all the premises have probabilities high enough,
the conclusions can't all have probabilities too low.

Adjunction

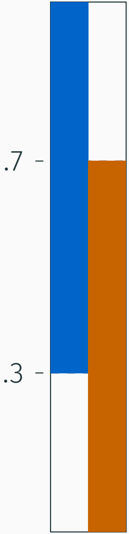


We have $p, q \models^{[.8,1]} p \wedge q$; when $\mathrm{P}(p), \mathrm{P}(q) \geq .8$, the lowest $\mathrm{P}(p \wedge q)$ can be is .6.

Still, $p, q, r, s \not\models^{[.8,1]} p \wedge q \wedge r \wedge s$, since when the premises all have probability .8, the conclusion can have probability as low as .2

$\models^{[.8,1]}$ lets us conjoin 2 or 3 sentences, but not 4 or more

Paraconsistency and reflexivity



Thresholds containing .5 give weakly paraconsistent, nonreflexive logics

For example, let $\mathrm{P}(p) = .5$ and $\mathrm{P}(q) = 0$.

Then $\mathrm{P}(p), \mathrm{P}(\neg p) \in (.3, 1]$ and $\mathrm{P}(p), \mathrm{P}(q) \in \overline{(.3, 1]}$, so:

- $p \not\models^{(.3,1]} p$
- $p, \neg p \not\models^{(.3,1]} q$

This achieves the intended 'classical-limit' behaviour:

Strength ordering

If $\beta \supseteq \alpha$, then $\models^{\beta} \subseteq \models^{\alpha}$

Classical at the top

$\models^{\{1\}}$ is Set-Set classical logic.
($\Gamma \models^{(0,1]} \Delta$ iff there is a contradiction in Γ or a tautology in Δ.)

No sudden leaps

For any closed α (including $\{1\}$), if $\Gamma \models^{\alpha} \Delta$,
there is some $\beta \supsetneq \alpha$
with $\Gamma \models^{\beta} \Delta$

Adjunctions:

Let P_n be the first n atomic sentences, and let $Conj_n$ be $[P_n \succ \bigwedge P_n]$.

Then $\vDash^\alpha$ validates $Conj_n$ iff $\alpha \subseteq (\frac{n}{n+1}, 1]$.

Modus ponenseses (Sorites in the medieval sense)

Let MP_n be $[p_1, p_1 \supset p_2, \ldots, p_{n-1} \supset p_n \succ p_n]$.

Then $\vDash^\alpha$ validates MP_n iff $\alpha \subseteq (\frac{n}{n+1}, 1]$.

Tighter standards support more interaction among formulas:

Size:

Let the size of the argument $[\Gamma \succ \Delta]$ be $|\Gamma| + |\Delta|$.

Then if $[\Gamma \succ \Delta]$ is classically valid and has size n, it is $\models^{(\frac{n-1}{n},1]}$-valid.

For any high set α:

Conditional right

If $\Gamma, A \vDash^{\alpha} B, \Delta$ then $\Gamma \vDash^{\alpha} A \supset B, \Delta$

Conjunction left

If $\Gamma, A, B \vDash^{\alpha} \Delta$, then $\Gamma, A \wedge B \vDash^{\alpha} \Delta$

Disjunction right

If $\Gamma \vDash^{\alpha} A, B, \Delta$, then $\Gamma \vDash^{\alpha} A \vee B, \Delta$

None of the above are invertible in general.

Negation left and right

$\Gamma, A \vDash^{\alpha} \Delta$ iff $\Gamma \vDash^{\alpha} \neg A, \Delta$,
and $\Gamma \vDash^{\alpha} A, \Delta$ iff $\Gamma, \neg A \vDash^{\alpha} \Delta$

Monotonicity

For any α, if $\Gamma \vDash^{\alpha} \Delta$, then $\Gamma, \Gamma' \vDash^{\alpha} \Delta, \Delta'$

Reflexivity

$\vDash^{\alpha}$ is reflexive iff $.5 \notin \alpha$

Transitivity

$\vDash^{\alpha}$ is transitive (in basically any form) iff either $.5 \in \alpha$ or $\alpha = \{1\}$

Only $\vDash^{\{1\}}$ is fully Tarskian.
Since we probably often want $.5 \notin \alpha$,
we should expect reflexive and nontransitive results.

Similarity

Given a vague predicate F, we assume a similarity predicate $\sim_F$.

When $x \sim_F y$, then x and y are similar enough to stand adjacent in a sorites series for F.

We suppose all similarity facts are fully certain:
probability 0 or 1

Similarity-respecting probability

P respects $\sim_F$ to within $\varepsilon \in [0, 1]$ iff
for all a, b, if $\mathrm{P}(a \sim_F b) = 1$, then $|\mathrm{P}(Fa) - \mathrm{P}(Fb)| \leq \varepsilon$.

That is, the probabilities of two F-similar things being F never differ from each other by more than ε

Symmetric consequence over similarity-respecting probability

$\Gamma \vDash^{\alpha}_{\varepsilon} \Delta$ iff:
there is no probability distribution P that respects $\sim_F$ to within ε
such that $\mathrm{P}(\gamma) \in \alpha$ for all $\gamma \in \Gamma$
and $\mathrm{P}(\delta) \in \overline{\alpha}$ for all $\delta \in \Delta$

Similarity-respecting probability

P respects $\sim_F$ to within $\varepsilon \in [0, 1]$ iff
for all a, b, if $\mathsf{P}(a \sim_F b) = 1$, then $|\mathsf{P}(Fa) - \mathsf{P}(Fb)| \leq \varepsilon$.

That is, the probabilities of two F-similar things being F
never differ from each other by more than ε

Symmetric consequence over similarity-respecting probability

$\Gamma \vDash^{\alpha}_{\varepsilon} \Delta$ iff:
there is no probability distribution P that respects $\sim_F$ to within ε
such that $\mathsf{P}(\gamma) \in \alpha$ for all $\gamma \in \Gamma$
and $\mathsf{P}(\delta) \in \overline{\alpha}$ for all $\delta \in \Delta$

1-step Tolerance

$Fa, a \sim_F b \vDash^{\alpha}_{\varepsilon} Fb$ iff

for all $x \in \alpha : \ x - \varepsilon \notin \overline{\alpha}$.

n-step Tolerance

$Fa_0, a_0 \sim_F a_1, \ldots, a_{n-1} \sim_F a_n \vDash^{\alpha}_{\varepsilon} Fa_n$ iff

for all $x \in \alpha : \ x - n\varepsilon \notin \overline{\alpha}$.

Not quantified tolerance tho

Suppose that $\varepsilon > 0$ and for all $x \in \alpha : \ x - \varepsilon \notin \overline{\alpha}$.
Then we have $\vDash^{\alpha}_{\varepsilon} (Fa \wedge a \sim_F b) \supset Fb$,
but $\nvDash^{\alpha}_{\varepsilon} \forall xy((Fx \wedge x \sim_F y) \supset Fy)$

Check this out:

Let Γ say that there is a 10-step sorites;
that is, let $\Gamma = \{Fa_0, \neg Fa_{10}\} \cup \{a_i \sim_F a_{i+1} \mid 0 \leq i < 10\}$.
By adjusting α, ε, we get different verdicts about Γ.

- If we don't have tolerance at all:
  nothing interesting seems to happen.
- If we have tolerance, but not yet 5-step tolerance:
  we get the usual kind of nontransitivity.
- If we have at least 10-step tolerance:
  no probability assignment can bring all of Γ into α,
  and so $\Gamma \vDash^{\alpha}_{\varepsilon} \Delta$ for any Δ.

Check this out:

Let Γ say that there is a 10-step sorites;
that is, let $\Gamma = \{Fa_0, \neg Fa_{10}\} \cup \{a_i \sim_F a_{i+1} \mid 0 \leq i < 10\}$.
By adjusting α, ε, we get different verdicts about Γ.

If we have at least 5-step tolerance but not yet 10-step,
Γ is not explosive, and an interesting phenomenon emerges.

Since a_5 is 5 steps from a_0,
if α, ε give us 5-step tolerance then we have $\Gamma \vDash^{\alpha}_{\varepsilon} Fa_5$.

But a_5 is also 5 steps from a_{10},
so in this case we also have $\Gamma \vDash^{\alpha}_{\varepsilon} \neg Fa_5$.

And since Γ is not explosive
(and everything is classical)
we have $\Gamma \not\models^{\alpha}_{\varepsilon} Fa_5 \wedge \neg Fa_5$.

The setup entails two things that are contradictory,
but it does not entail any contradiction.

(A nontransitive echo
of nonreflexive weak paraconsistency)

Conclusion

- We can understand vague predicates through probability distributions on thresholds.
- Preserving high probability is not well-behaved, but symmetric consequence shows how valid reasoning can approach classical logic as probabilistic standards tighten.
- Adding similarity lets us validate longer and longer chains of soritical reasoning as standards tighten.
- The result is weakly inconsistent, as good as we’re going to get while staying classical.