Uncut

David Ripley

University of Connecticut
http://davewripley.rocks

The questions

Paradoxes

Harry is an old dog.

He wasn't always old.

But a dog doesn't just get old in a second:
if Harry wasn't old at t, and $|t - u| < 1$ second,
he wasn't old at u either.

From this, we can prove he isn't old now.

But it doesn’t work; he remains old.

Something’s gone wrong in the proof.

Or consider: ‘If this sentence is true, then Harry isn’t old’.

If this sentence is indeed true, then if it’s true, then Harry isn’t old.
And of course, if it’s true then it’s true.

So if it’s true, then Harry isn’t old.

That is to say, it’s true!

So Harry isn’t old.

This doesn’t work any better than the last proof.

Harry is still old.

The questions

Meaning

What has gone wrong in the reasoning?

One problem:
the steps seem to be guaranteed
by the meanings of the involved vocabulary.

If ‘old’ drew a sharp boundary,
it would mean something different from what it means.

If ‘if…then’ didn’t validate modus ponens,
it would mean something different from what it means.

But then the first argument seems to come out in the clear,
as a matter of these meanings.

If ‘X is true’ were not interchangeable with X,
‘true’ would mean something different from what it means.

If ‘If…then’ didn’t validate modus ponens,
including under a conditional antecedent,
or if ‘If X then X’ failed,
it would mean something different from what it means.

But then the second argument is in the clear as well,
as a matter of these meanings.

Most approaches to these paradoxes fall into one of two camps:

1. Avoid. Design another language,
or restrict the ones we've got to cases where these don't arise.

2. Reject the claims about meaning.
Some step of these arguments fails.
So 'old' or 'true' or 'if…then' have non-obvious meanings.

It would be nice if there were another way.

Maybe everything means just what it seems to,
and yet the paradoxical arguments really are no good.

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After all, every step in the arguments seems required by the meanings involved.

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seems required by the meanings involved.

The target must be:
the way these steps are combined.

Bounds

Positions

How to understand meanings?

In terms of bounds on positions.

In conversation, we need to keep track of the claims participants make.

“You’re wrong; I didn’t eat it” is only appropriate to assert if you said I ate it, at least implicitly.

We keep track of each other's positions.

Formally, a position is $[\Gamma \succ \Delta]$,
with Γ, Δ finite sets of sentences.

Γ are those things asserted,
and Δ those things denied.

There are a wide range of norms
governing our assertions and denials.

They can be wise or foolish,
justified or reckless,
kind or cruel,
true or false,
and so on.

The bounds I am interested in are to do with fit.

A position fits together with itself or not.

Positions that do fit together are in bounds;
others are out of bounds.

For example, the position that asserts
“Melbourne is bigger than Brisbane”
and “Brisbane is bigger than Darwin”

while denying
“Melbourne is bigger than Darwin”

is out of bounds.

We can give meanings of vocabulary
in terms of interactions with the bounds.

Bounds

Consequence

A starting point:

$[A \succ A]$ is always out of bounds, no matter what A is.

If $[\Gamma \succ \Delta]$ is out of bounds, so is $[\Gamma, \Gamma' \succ \Delta, \Delta']$.

In sequent-calculus form:

$$\text{Id: } \frac{}{[A \succ A]} \qquad \text{D: } \frac{[\Gamma \succ \Delta]}{[\Gamma, \Gamma' \succ \Delta, \Delta']}$$

Whether a position is in or out of bounds
just is the question whether an argument is invalid or valid,
at least in one worthwhile sense of ‘valid’.

The strategy:

Give sound proof systems for the bounds,
capturing enough vocabulary for the paradoxes,
and see if any trouble arises.

Id and D apply no matter what vocabulary is in play.

Do any other rules?

There is a usual candidate: Cut.

$$\text{Cut:} \quad \frac{[\Gamma \succ \Delta, A] \qquad [A, \Gamma \succ \Delta]}{[\Gamma \succ \Delta]}$$

This says that if $[\Gamma \succ \Delta]$ is in bounds,
then so is at least one of $[\Gamma \succ \Delta, A]$ and $[A, \Gamma \succ \Delta]$.

If a position is in bounds,
there is always something it can say about A, for any A.

According to cut, there are no double-binds.

If we cannot assert A and we cannot deny it, our position already conflicts with itself.

This is at least less obvious
than Id and D.

Perhaps we can get by without it.

Say a position is committed to something
iff its denial can't be in-bounds added.

If Cut fails, there is a difference
between asserting something (even implicitly)
and being committed to it.

Vocabulary

UFO vocabulary

Theories of vocabulary meaning in these terms
look a lot like normal proof theories.

(The proofs themselves have no semantic import;
they just tell us about the bounds.)

Here is a theory for the Usual First Order vocabulary.

$$\neg\text{L:}\quad \frac{[\Gamma \succ \Delta, A]}{[\neg A, \Gamma \succ \Delta]} \qquad \neg\text{R:}\quad \frac{[A, \Gamma \succ \Delta]}{[\Gamma \succ \Delta, \neg A]}$$

$$\wedge\text{L:}\quad \frac{[A, B, \Gamma \succ \Delta]}{[A \wedge B, \Gamma \succ \Delta]} \qquad \wedge\text{R:}\quad \frac{[\Gamma \succ \Delta, A] \qquad [\Gamma \succ \Delta, B]}{[\Gamma \succ \Delta, A \wedge B]}$$

$$\forall\text{L:}\quad \frac{[A(t), \Gamma \succ \Delta]}{[\forall x A(x), \Gamma \succ \Delta]} \qquad \forall\text{R:}\quad \frac{[\Gamma \succ \Delta, A(a)]}{[\Gamma \succ \Delta, \forall x A(x)]}$$

$$=\text{-drop:}\quad \frac{[t = t, \Gamma \succ \Delta]}{[\Gamma \succ \Delta]} \qquad =\text{-sub:}\quad \frac{[t = u, \Gamma(t) \succ \Delta(t)]}{[t = u, \Gamma(u) \succ \Delta(u)]}$$

This gives classical logic
over the full vocabulary.

Vocabulary

Vague predicates

Vagueness is about the relation between two predicates:
the vague predicate *P*, and *P*-similarity.

If two things are *P*-similar,
then if one is *P* so is the other.

Rules for $\sim_P$:

$$\sim\text{-drop:}\quad \frac{[t \sim_P t, \Gamma \succ \Delta]}{[\Gamma \succ \Delta]}$$

$$\sim\text{-sym:}\quad \frac{[t \sim_P u, \Gamma \succ \Delta]}{[u \sim_P t, \Gamma \succ \Delta]} \qquad \sim\text{-sym:}\quad \frac{[\Gamma \succ \Delta, t \sim_P u]}{[\Gamma \succ \Delta, u \sim_P t]}$$

and tolerance:

$$\text{Tol:}\quad \frac{[\Gamma \succ \Delta, t \sim_P u]}{[Pt, \Gamma \succ \Delta, Pu]}$$

Positions like $[Pt, t \sim_P u \succ Pu]$
and $[\succ \forall xy(Px \land x \sim_P y \to Py)]$
are out of bounds.

But sorites series are not.

Let $\Sigma = \{a_1 \sim_P a_2, \ldots, a_{n-1} \sim_P a_n\}$.
So long as $n > 2$, we cannot rule out $[\Sigma, Pa_1 \succ Pa_n]$.

(This is where models come in.)

With Cut, this would not be possible.

If we have both $[\Sigma, Pa_1 \succ Pa_i]$ and $[\Sigma, Pa_i \succ Pa_{i+1}]$, by cut we would get $[\Sigma, Pa_1 \succ Pa_{i+1}]$.

But tolerance gives us these!

If we assume Cut, then,
tolerance causes its trouble.

Positions recording the existence of a sorites series
are out of bounds.

So don’t assume Cut!
Tolerance, and so vague meanings, are fine.

Sorites series exist,
and there is nothing wrong with saying so.

We must allow that there are double binds.

We cannot fully categorize as *P* or not
each member of a sorites series.

We are committed to a contradiction.

Vocabulary

Semantic predicates

As far as the bounds are concerned,
truth can be idle:

$$T\langle\rangle\mathrm{L}: \quad \frac{[A, \Gamma \succ \Delta]}{[T\langle A\rangle, \Gamma \succ \Delta]} \qquad T\langle\rangle\mathrm{L}: \quad \frac{[\Gamma \succ \Delta, A]}{[\Gamma \succ \Delta, T\langle A\rangle]}$$

Let $\kappa_A = T\langle\kappa_A\rangle \to A$.

$$
T\langle\rangle\text{L: } \frac{[T\langle\kappa_A\rangle, T\langle\kappa_A\rangle \to A \succ A]}{\text{which is: } \dfrac{[T\langle\kappa_A\rangle, T\langle\kappa_A\rangle \succ A]}{[T\langle\kappa_A\rangle \succ A]}}
\qquad
\to\text{R: } \frac{[T\langle\kappa_A\rangle \succ A]}{T\langle\rangle\text{R: } \dfrac{[\succ T\langle\kappa_A\rangle \to A]}{[\succ T\langle\kappa_A\rangle]}}
$$

By cutting $[T\langle\kappa_A\rangle \succ A]$ with $[\succ T\langle\kappa_A\rangle]$,
we get $[\succ A]$.

That's bad!

Again, though, with no Cut there is no problem.

The truth rules are even conservative.

(Again, models come in handy here.)

Truth can be as simple as it seems,
and paradoxical sentences can exist just fine.

The bounds rule out either asserting or denying them.

We are again committed to a contradiction.

The same goes for talk about the bounds themselves.

$$OB\langle\rangle\text{D}: \quad \frac{}{[\Gamma, OB\langle \Gamma \succ \Delta \rangle \succ \Delta]} \qquad OB\langle\rangle\text{R}: \quad \frac{[\Gamma, \Gamma_B \succ \Delta_B, \Delta]}{[\Gamma_B \succ \Delta_B, OB\langle \Gamma \succ \Delta \rangle]}$$

The usual validity paradoxes also rely on Cut
to cause any trouble.

Conclusion

- Paradoxes seem to be guaranteed by the meanings of the involved vocabulary.
- But this need not be so.
- Meaning is a matter of bounds on positions, and these bounds do not obey Cut.
- Everything (else) is just as it seems.