One step is enough

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ST

Propositional language with $\neg, \wedge, \vee, \top, \bot$

Strong kleene models with values $\{1, \frac{1}{2}, 0\}$:

$$
\begin{array}{rcl}
[\![\neg A]\!] & \text{is} & 1 - [\![A]\!] \\
[\![A \wedge B]\!] & \text{is} & \min([\![A]\!], [\![B]\!]) \\
[\![A \vee B]\!] & \text{is} & \max([\![A]\!], [\![B]\!]) \\
[\![\top]\!] & \text{is} & 1 \\
[\![\bot]\!] & \text{is} & 0
\end{array}
$$

‘2-valued models’ are models that only use the values $\{1, 0\}$.

If all atoms are 2-valued, the whole model is.
These are ordinary Boolean valuations.

An inference is a pair of sets of sentences (premises and conclusions).
A consequence relation is a set of inferences (valid ones).

Use a class of models to determine a consequence relation
by giving a counterexample relation between models and inferences.

The valid inferences are the ones with no counterexample.

Focus on mixed consequence:
(See Chemla & Égré 2019 in RSL)

Given sets P and C of values,
a model is a PC counterexample to an argument $[\Gamma \succ \Delta]$ iff
it assigns everything in Γ into P and nothing in Δ into C.

I'll write $\vdash^{PC}$ for the consequence relation so determined.

When $\Gamma \vdash^{PC} \Delta$,
in any model where all the Γs are P, some Δ is C.

Let $s = \{1\}$ and $s = \{1, \frac{1}{2}\}$.

Then strong kleene logic (K3) is $\vdash^{ss}$,
and LP is $\vdash^{tt}$.

But we also have $\vdash^{st}$ and $\vdash^{ts}$.

If we restrict to two-valued models, we lose the distinction between s and t.

A two-valued model is a CL-counterexample to an inference $[\Gamma \succ \Delta]$ when it assigns 1 to everything in Γ and 0 to everything in Δ.

$\vdash_{cl}$ is the set of inferences without CL-counterexamples.

$\vdash^{ts}$ is a weird beast.

Not much at all is $\vdash^{ts}$-valid:
just things like $[\bot \succ]$, $[\succ \top]$ and the like.

(Note that $p \nvdash^{ts} p$)

But the notion of a *ts* counterexample has an important role to play later.

$$\vdash^{st} = \vdash_{cl}$$

That is, $\Gamma \vdash^{st} \Delta$ iff $\Gamma \vdash_{cl} \Delta$

We have specified the same set of inferences two different ways.

This difference in specification matters when we remove models.

Let a model be transparent when $[\![T\langle A\rangle]\!] = [\![A]\!]$ for every sentence A.

There are no transparent 2-valued models,
so no transparent counterexamples to, say, $[p \succ q]$.

But there are transparent models galore,
including such counterexamples.

Because of this, $\vdash^{st}$ can be conservatively extended with a transparent truth predicate.

Since $\vdash^{st}$ is $\vdash_{cl}$, this means that $\vdash_{cl}$ can be conservatively extended with a transparent truth predicate.


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CL
LP
ST

The resulting extension $\vdash^{stT}$, however, is nontransitive: it is not closed under cut.

$$\text{Cut:} \quad \frac{[\Gamma \succ \Delta, A] \qquad [A, \Gamma \succ \Delta]}{[\Gamma \succ \Delta]}$$

Where λ is a liar sentence,

$\vdash^{stT} \lambda$ and $\lambda \vdash^{stT}$, but $\nvdash^{stT}$

So: $\vdash^{st}$ and $\vdash_{cl}$ are identical, and both transitive.

But if we restrict our models and proceed via CL counterexamples, we only reach transitive consequence relations.

This makes some people happy,
but is a disaster for transparent truth, vagueness, etc.

Otoh, if we restrict our models and proceed via *st* counterexamples, we can reach nontransitive consequence relations.

Metainferences

Cut is an example of a metainference.

A metainference is a set of premise inferences and a conclusion inference.

Two kinds of metainferential validity are relevant here:

Global

A metainference $[\Gamma_1 \succ \Delta_1], \ldots, [\Gamma_n \succ \Delta_n] \Rightarrow [\Gamma \succ \Delta]$ is globally valid iff:
if there is a counterexample to $[\Gamma \succ \Delta]$,
then there is a counterexample to some $[\Gamma_i \succ \Delta_i]$

Local

A metainference $[\Gamma_1 \succ \Delta_1], \ldots, [\Gamma_n \succ \Delta_n] \Rightarrow [\Gamma \succ \Delta]$ is locally valid iff:
each model that is a counterexample to $[\Gamma \succ \Delta]$
is itself a counterexample to some $[\Gamma_i \succ \Delta_i]$

If two notions of model and counterexample agree
on which inferences are valid,
then they agree on which metainferences are globally valid.

Eg cut is globally valid for models with *st* counterexamples,
as it is for two-valued models with CL counterexamples.

These determine the same set of inferences,
and the set is closed under cut.

Local metainferential validity is more sensitive.

Cut is locally valid for two-valued models with CL counterexamples, but not for models with *st* counterexamples.

$$\text{Cut:} \quad \frac{[\Gamma \succ \Delta, A] \qquad [A, \Gamma \succ \Delta]}{[\Gamma \succ \Delta]}$$

This holds even for our simple propositional language; no truth predicate or other funny business is needed.

Local and global validity of metainferences might remind you of derivability and admissibility for rules in a proof system:

- Global/admissible supervenes on which inferences are valid; local/derivable is more sensitive to details of the models/proofs.
- Local/derivable implies global/admissible; not vice versa.
- For 0-premise metainferences, the converse holds as well.
- Local/derivable is preserved on restricting models/adding rules; global/admissible is not.

And indeed, there is a match between global and admissible, given soundness and completeness:

If a proof system is sound and complete for a model system, then a metainference is admissible in the proof system iff it's valid in the model system.

But no such connection holds for local and derivable; these are in general independent statuses. (See Humberstone 1995 in JPL.)

A metainferential hierarchy

$\vdash^{tt}$ matches $\vdash_{cl}$ on logical truths,
but not on which inferences are valid.
$\vdash^{st}$ takes an extra step, matching $\vdash_{cl}$ on inferences as well.

ST models match CL models on which inferences are valid,
but not on which metainferences are locally valid.

Can we take an extra step,
matching CL models on (local) metainferences as well?

Thanks to the Buenos Aires Logic Group, now we can.

Pailos 2019a and b in JANCL and RSL
Barrio, Pailos, Szmuc 2019a and b in JPL and Synthese
Da Ré, Pailos, Szmuc, Teijeiro in progress (?)

(See also Scambler 2019 in JPL)

The key is to look into the definition of local metainferential validity:

Local

A metainference $[\Gamma_1 \succ \Delta_1], \ldots, [\Gamma_n \succ \Delta_n] \Rightarrow [\Gamma \succ \Delta]$ is locally valid iff:
each model that is a counterexample to $[\Gamma \succ \Delta]$
is itself a counterexample to some $[\Gamma_i \succ \Delta_i]$

We have multiple uses of ‘counterexample’ in play.
What if we mix them?

The key is to look into the definition of local metainferential validity:

TS/ST Local

A metainference $[\Gamma_1 \succ \Delta_1], \ldots, [\Gamma_n \succ \Delta_n] \Rightarrow [\Gamma \succ \Delta]$ is locally valid iff:
each model that is an st counterexample to $[\Gamma \succ \Delta]$
is itself a ts counterexample to some $[\Gamma_i \succ \Delta_i]$

We have multiple uses of ‘counterexample’ in play.
What if we mix them?

Just as *s* is a stricter standard than *t*,
so *ts* is a stricter standard than *st*.

TS/ST is a set of metainferences:
a metainferential analog of $\vdash^{st}$.

As it turns out, a metainference is locally valid in CL models iff it is TS/ST valid.

So TS/ST matches CL models ‘up a level’.

Let an inference $\mathcal{I}$ be TS/ST valid iff
the metainference $\Rightarrow \mathcal{I}$ is TS/ST valid.

Then this is $\vdash^{st}$, which we know matches $\vdash_{cl}$.

So TS/ST models match CL models on inferences just like ST models,
plus match for metainferences as well.


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LP
ST
TS/ST

Here we go!

A meta^{0}inference is an inference;

a meta^{n+1} inference is a set of premise meta^{n}inferences and a conclusion meta^{n}inference.

TS/ST Local

A metainference $[\Gamma_1 \succ \Delta_1], \ldots, [\Gamma_n \succ \Delta_n] \Rightarrow [\Gamma \succ \Delta]$ is locally valid iff:
each model that is an st counterexample to $[\Gamma \succ \Delta]$
is itself a ts counterexample to some $[\Gamma_i \succ \Delta_i]$

In other words:

A model is a TS/ST counterexample to a metainference
$[\Gamma_1 \succ \Delta_1], \ldots, [\Gamma_n \succ \Delta_n] \Rightarrow [\Gamma \succ \Delta]$ iff:
it is an *st* counterexample to $[\Gamma \succ \Delta]$,
but not a *ts* counterexample to any $[\Gamma_i \succ \Delta_i]$

A T_1 counterexample to an inference is an *st* counterexample;
an S_1 counterexample to an inference is a *ts* counterexample.

A T_{n+1} counterexample to a meta^{n+1}inference is a model that is
a T_n counterexample to the conclusion meta^ninference
but not an S_n counterexample to any premise meta^ninference.

An S_{n+1} counterexample to a meta^{n+1}inference is a model that is
an S_n counterexample to the conclusion meta^ninference
but not a T_n counterexample to any premise meta^ninference.

A CL_ω counterexample to an inference is a CL counterexample.

A CL_ω counterexample to a meta^{n+1}inference is a model that is a CL_ω counterexample to the conclusion meta^{n}inference but not a CL_ω counterexample to any premise meta^{n}inference.

An st_ω counterexample to an inference is an st counterexample.

An st_ω counterexample to a meta^{n+1}inference is a model that is an st_ω counterexample to the conclusion meta^{n}inference but not an st_ω counterexample to any premise meta^{n}inference.

Say a meta^{n}inference is T_n valid iff it has no T_n counterexample.

Say a meta^{n}inference $\mathcal{I}$ is T_{n+1} valid iff $\Rightarrow \mathcal{I}$ is T_{n+1} valid.

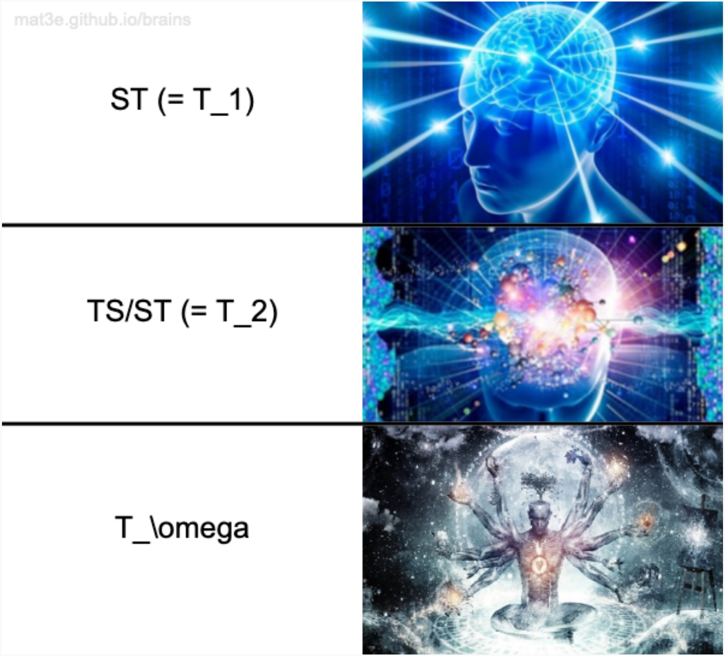
Then we have: for $m \geq n$,
T_m and T_n agree on validity for meta^{n}inferences.

And T_n and CL_ω agree on validity for meta^{n}inferences.

Say that a meta^{n}inference is T_ω valid iff
it is T_n valid (and therefore T_m valid for all $m \geq n$).

Then T_ω and CL_ω agree on validity for meta^{n}inferences for all n.

But since T_ω is defined over all models,
it allows for conservative extension with transparent truth.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ST (= T_1)
TS/ST (= T_2)
T_\omega

A challenge to ST

"Non-classical theories of truth pursue two conflicting desiderata. On the one hand, they search for a paradox-free transparent truth predicate. On the other hand, they want to retain as much classical logic as possible.... Thus, though it might be argued that ST seems to do much better than the other inferential non-classical solutions to paradoxes—precisely because it resolves paradoxes while 'mutilating' less classical logic than the other non-classical theories, TS/ST seems to work even better than ST. TS/ST retains every classically valid inference, as ST does, but, moreover, it recovers every classically valid metainference—while ST loses Cut (and many other classically valid metainferences)." (Pailos 2019, emphasis added)

"[T]he proponent of logics like [ST] as solutions to the paradoxes faces some difficult questions. First, they must say whether or not they mean to generalize their view to higher finite levels. If they don't, they must explain why the 'more classical logic is better' line of thought above is misguided." (Scambler 2019)

"It seems to me that if Ripley's use of [ST] is attractive, one can make a case that each theory T_n for $n > 1$ is still more attractive, because it gets us more classical logic. If it was a good idea to expand the horizons of classicality from mere [LP] to [ST], why isn't it good to have…the theory T_2, pushing back the boundaries of non-classicality to the third level…?" (Scambler 2019, emphasis added)

In posing these challenges to ST,
Pailos and Scambler both seem to endorse two views:

First, that ‘more classical logic is better’;
second, that the T_i s get ‘more classical’ as i increases.

If these claims are both correct,
then any T_i occupies an unstable position;
T_{i+1} is better.

Many of us have no particular desire to be classical
for classicality's sake.

Classical logic has had detractors for as long as it has existed;
and although it gained a certain sort of hegemonic status
in analytic philosophy in the late 20th century,
that moment is passing.

Classical logic is an inheritance we’ve received,
not a goal we’re aiming for.

It’s up to us to figure out whether and how to use it
to reach our actual goals.

Here is a theory of how conjunction and negation interact with coherent patterns of assertion and denial:

It’s coherent to assert $A \wedge B$ iff it’s coherent to assert both A and B.
It’s coherent to deny $A \wedge B$ iff it’s coherent to deny one of them.

It’s coherent to assert/deny $\neg A$ iff it’s coherent to deny/assert A.

Now, let $\Gamma \vdash \Delta$ mean that it's incoherent to assert everything in Γ while denying everything in Δ.

Suppose as well that $\Gamma \vdash \Delta$ whenever Γ and Δ overlap.

It follows from all this that every classically-valid inference is in $\vdash$.

These are all contestable suppositions,
but someone who accepts them would have a reason
to accept classical logic in this sense.

What about cut?

$$\text{Cut:} \quad \frac{[\Gamma \succ \Delta, A] \qquad [A, \Gamma \succ \Delta]}{[\Gamma \succ \Delta]}$$

This says: if a collection of acts is coherent,
then either it's coherent to extend it with a denial of A,
or it's coherent to extend it with an assertion of A.

It's a 'no-double-binds' requirement on coherence.

But perhaps you can be in a double bind,
and still be coherent.

Certainly it fits with all our foregoing suppositions
to allow this.

Such a view would accept all classical inferences,
but reject cut.

So TS/ST and T_ω match CL_ω in validating cut,
while st_ω does not.

From the present point of view, this is not a drawback of st_ω,
but rather a reason to think it's getting something right,
more right than TS/ST or T_ω.

Where we have reason to think CL_ω gets things wrong,
disagreeing with it is no vice in a theory.

T_ω matches CL_ω on meta^{n}inferences, for every n.

And, since T_ω is based on three-valued models,
we can add transparent truth to it conservatively, reaching TT_ω.

This invalidates nothing: all CL_ω-valid meta^{n}inferences
are still valid in TT_ω.

Something strange has happened, though.
$\vdash^{TT_\omega} \lambda$ and $\lambda \vdash^{TT_\omega}$ and $\nvdash^{TT_\omega}$,
and yet $\vdash^{TT_\omega} [\succ\lambda], [\lambda\succ] \Rightarrow [\succ]$

TT_ω contains cut, but does not obey it.
(The same goes for TS/ST.)

The phenomenon is repeated at every level.

Say that a meta^{n}inference $\mathcal{I}_1, \ldots, \mathcal{I}_n \Rightarrow \mathcal{I}$ is obeyed iff either $\mathcal{I}$ is valid, or some $\mathcal{I}_i$ is not valid.

Then for any n, TT_ω contains some meta^{n}inference that it does not obey.

Does this matter?
It depends on what application we have in mind.

Is there an interpretation of meta^{n}inferences
that makes this the right result? Maybe.

If we care about coherence constraints on assertions and denials,
then we apply all this at the level of inferences.

Here, we should care about metainferences
insofar as they express connections between inferences.

And if they are not obeyed, they do not do this.

st_{ω}, by contrast, obeys every meta^{n}inference
that it contains, for $n \geq 1$.

Moreover, if we focus only on inferences,
there is no question of approximation to $\vdash_{cl}$.

$\vdash^{st}$ and $\vdash^{st_\omega}$ are exactly the same,
as are $\vdash^{stT}$ and $\vdash^{stT_\omega}$

$\vdash^{st} = \vdash^{st_\omega} = \vdash_{cl}$,
but $\vdash^{stT}$ and $\vdash^{stT_\omega}$ are nothing like $\vdash^{clT}$,
and a good thing too!

Where meta^{n}inferences matter,
there the difference between st_ω and T_ω might ramify.

But if they just matter for their connections to inferences,
it's st_ω that gets things right, not T_ω.

Conclusion

- The technology of meta^{n}inferences allows us to raise and explore subtle questions about models and counterexamples.
- The hierarchy of T_ns and its limit T_ω generalize *st*-like phenomena to all metainferential levels.
- Classical logic is not an ideal to be aimed for, but rather an influential and sometimes-useful family of ideas.
- When our focus is on inferences, we should look at which metainferences are obeyed.
- For exploring constraints on coherent assertion & denial, our focus should be on inferences.