General elimination isn't general enough

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Introductions as definitions

Gentzen 1934:

"The introductions represent, as it were, the 'definitions' of the symbols concerned, and the eliminations are…the consequences of these definitions.

$$\wedge\text{I}: \quad \frac{A \qquad B}{A \wedge B} \qquad\qquad \vee\text{I}_l: \quad \frac{A}{A \vee B} \qquad \vee\text{I}_r: \quad \frac{B}{A \vee B}$$

$$\wedge\text{E}_l: \quad \frac{A \wedge B}{A} \qquad \wedge\text{E}_r: \quad \frac{A \wedge B}{B} \qquad\qquad \vee\text{E}: \quad \frac{A \vee B \qquad \begin{matrix}[A] \\ \vdots \\ C\end{matrix} \qquad \begin{matrix}[B] \\ \vdots \\ C\end{matrix}}{C}$$

"Such [introduction] laws will be 'self-justifying': we are entitled simply to stipulate [them], because by so doing we fix...the meanings of the logical constants that they govern"

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General elimination

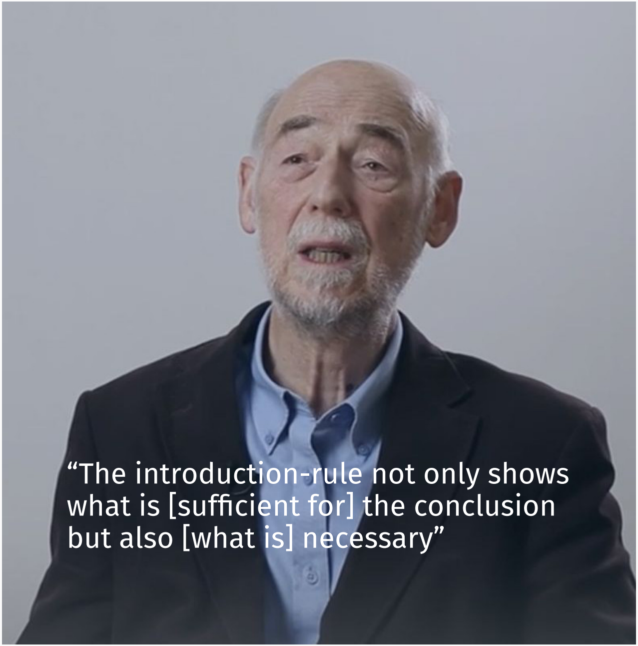
Question:

Given introduction rules for a connective,

what elimination rules are justified?

"[W]hatever follows from the sufficient grounds for deriving a formula must follow from that formula...

[W]hatever follows from a formula must follow from the sufficient grounds for deriving the formula"


"The introduction-rule not only shows
what is [sufficient for] the conclusion
but also [what is] necessary"

Moriconi & Tesconi:

“[I]t is quite natural to ask what consequences can be drawn from A, given that A can be produced *only* by [certain] rules. The answer is: we can draw all the consequences that we can draw from the premisses of those rules”

$$\vee \mathrm{I}_l: \quad \frac{A}{A \vee B} \qquad \vee \mathrm{I}_r: \quad \frac{B}{A \vee B}$$

$$\vee \mathrm{GE}: \quad \frac{A \vee B \qquad \begin{array}{c}[A]\\ \vdots \\ C\end{array} \qquad \begin{array}{c}[B]\\ \vdots \\ C\end{array}}{C}$$

$$\wedge\text{I:} \quad \frac{A \qquad B}{A \wedge B}$$

$$\wedge\text{GE:} \quad \frac{A \wedge B \qquad \begin{array}{c} \underbrace{[A], [B]} \\ \vdots \\ C \end{array}}{C}$$

Local soundness and completeness

Pfenning & Davies coin ‘local soundness’ and ‘local completeness’ for certain often-desired properties of rules.

Local soundness means that all local peaks are reducible.

$$\wedge\text{GE}: \dfrac{\wedge\text{I}: \dfrac{\begin{array}{cc}\vdots & \vdots\\ A & B\end{array}}{A \wedge B} \qquad \begin{array}{c}\underbrace{[A],[B]}\\ \vdots \\ C\end{array}}{C} \quad\rightsquigarrow\quad \begin{array}{c}\begin{array}{cc}\vdots & \vdots\end{array} \\ \underbrace{A, \quad B} \\ \vdots \\ C\end{array}$$

GE guarantees this.

When rules are locally sound,
the elimination rules are not “too strong”

Anything that can be proved by I and then E
can be proved directly

(Tonk famously violates local soundness)

Given I rules for *A*, we can justify locally sound E rules for *A*:

Dummett again:

"The strategy...is that of all proof-theoretic justifications, namely, to show that we can dispense with the rule up for justification: if we have a valid argument for the premises of a proposed application of it, we already have a valid argument, not appealing to that rule, for the conclusion."

Local completeness means "we can apply the elimination rules to a judgment to recover enough [to reintroduce] the original judgment":

$$\wedge \mathrm{I}: \dfrac{\wedge\mathrm{GE}^1: \dfrac{A \wedge B \qquad [A]^1}{A} \qquad \wedge\mathrm{GE}^2: \dfrac{A \wedge B \qquad [B]^2}{B}}{A \wedge B}$$

GE guarantees this, too.

When rules are locally complete,
the elimination rules are not “too weak”

The elimination rules yield enough
to introduce the formula again

Local completeness is maybe not required for justifying E rules, but it pushes us to justify the strongest rules we can.

The list modality

I’m going to argue that GE
is not always the best way to get elimination rules.

The argument proceeds by way of an example:
a unary modality $\star$, given by its introduction rules.

The GE rule for $\star$ is ok,
but we can do better.

$$\star I_{\text{nil}}: \quad \frac{}{A^\star} \qquad\qquad \star I_{\text{cons}}: \quad \frac{A \qquad A^\star}{A^\star}$$

$\star\mathrm{I}_{\mathrm{nil}}$: $\dfrac{}{A^\star}$ $\star\mathrm{I}_{\mathrm{cons}}$: $\dfrac{A \qquad A^\star}{A^\star}$

$\star\mathrm{I}_{\mathrm{cons}}$ might seem redundant in two separate ways:

- $\star\mathrm{I}_{\mathrm{nil}}$ already proves $A^\star$ for all A
- We can't use $\star\mathrm{I}_{\mathrm{cons}}$ to prove anything we haven't already proved

$$
\to\mathrm{I}^1:\ \dfrac{\star\mathrm{I}_{\mathrm{cons}}:\ \dfrac{\wedge\mathrm{I}:\ \dfrac{[p]^1 \qquad \star\mathrm{I}_{\mathrm{nil}}:\ \dfrac{}{q^\star}}{p \wedge q^\star} \qquad \star\mathrm{I}_{\mathrm{nil}}:\ \dfrac{}{(p \wedge q^\star)^\star}}{(p \wedge q^\star)^\star}}{p \to (p \wedge q^\star)^\star}
$$

$\star I_{cons}$ makes a real difference, by adding many more proofs.
They're just not proofs *of* anything new.

If all we care about is provability rather than proofs,
then $\star I_{cons}$ is indeed redundant.

But if all we care about is provability,
we should be off this boat a long way back!

For the curious, though:

An argument has a proof in this system iff
replacing all $A^\star$s with $\top$ has a proof
in the base system.

The same will be true after we add elimination rules as well.

Say that a proof is canonical when:
it ends in an introduction rule
and its immediate subproofs are canonical.

Then a canonical proof of $A \wedge B$ is a pair
of a canonical proof of A and a canonical proof of B.

This way lies the BHK intepretation
and the Curry-Howard correspondence.

Canonical proofs of $A^\star$ are finite lists of canonical proofs of A:

$$\star\mathrm{I}_{\mathrm{nil}}:\quad \frac{}{A^\star} \qquad\qquad \star\mathrm{I}_{\mathrm{cons}}:\quad \frac{\begin{array}{c}\vdots\\ A\end{array} \qquad \star\mathrm{I}_{\mathrm{nil}}:\ \dfrac{}{A^\star}}{A^\star}$$

$$\star\mathrm{I}_{\mathrm{cons}}:\quad \frac{\begin{array}{c}\vdots\\ A\end{array} \qquad \star\mathrm{I}_{\mathrm{cons}}:\ \dfrac{\begin{array}{c}\vdots\\ A\end{array} \qquad \star\mathrm{I}_{\mathrm{nil}}:\ \dfrac{}{A^\star}}{A^\star}}{A^\star}$$

I’ve chosen lists as a simple example.

But the points to follow make sense
for inductively defined data types in general.

General elimination for lists

General elimination for $\star$:

$$\star\text{GE:}\quad \frac{A^\star \qquad C \qquad \begin{matrix}\underbrace{[A], [A^\star]} \\ \vdots \\ C\end{matrix}}{C}$$

It is locally sound:

$$\star\mathrm{I}_{\mathrm{nil}}:\ \star\mathrm{GE}:\ \dfrac{\dfrac{}{A^\star} \qquad \begin{matrix}\vdots \\ C\end{matrix} \qquad \begin{matrix}\overbrace{[A],[A^\star]} \\ \vdots \\ C\end{matrix}}{C} \quad \rightsquigarrow \quad \begin{matrix}\vdots \\ C\end{matrix}$$

$$\star\mathrm{I}_{\mathrm{cons}}:\ \star\mathrm{GE}:\ \dfrac{\dfrac{\begin{matrix}\vdots \\ A\end{matrix} \qquad \begin{matrix}\vdots \\ A^\star\end{matrix}}{A^\star} \qquad \begin{matrix}\vdots \\ C\end{matrix} \qquad \begin{matrix}\overbrace{[A],[A^\star]} \\ \vdots \\ C\end{matrix}}{C} \quad \rightsquigarrow \quad \begin{matrix}\overbrace{\begin{matrix}\vdots & & \vdots \\ A, & & A^\star\end{matrix}} \\ \vdots \\ C\end{matrix}$$

and locally complete:

$$\star\mathrm{GE}: \quad \dfrac{A^\star \qquad \star\mathrm{I}_{\mathrm{nil}}: \ \dfrac{}{A^\star} \qquad \star\mathrm{I}_{\mathrm{cons}}: \ \dfrac{[A] \qquad [A^\star]}{A^\star}}{A^\star}$$

Local soundness and completeness
mean that we can justify $\star$GE by appeal to the I rules,
and that $\star$GE lets us draw all the conclusions justifiable in this way.

What more could we want?

If all we cared about was provability, we could stop.

But what we care about is proofs.

Local completeness tells us we have some proof
of every justifiable conclusion.
But what if we want every justifiable proof?

Given what finite lists are,
certain operations on them make sense.

For example, given a finite list π of proofs of *A* and a proof α of *A*, we can determine a particular proof of *A* as follows:

If π is the empty list, we use α, and if π is not empty, we use its head, its first element.

$\star$GE lets us form a proof that 'does' this:

$$\star\text{GE:}\quad \frac{\begin{array}{c}\pi\\ A^\star\end{array} \qquad \begin{array}{c}\alpha\\ A\end{array} \qquad [A]}{A}$$

And given two finite lists π, ρ of proofs of A,
we can determine a particular finite list of proofs of A as follows:

If π is the empty list, we use ρ,
and if π is not empty, we use its tail.

$\star$GE lets us form a proof that does this too:

$$\star\text{GE:}\quad \frac{\begin{matrix}\pi \\ A^\star\end{matrix} \qquad \begin{matrix}\rho \\ A^\star\end{matrix} \qquad [A^\star]}{A^\star}$$

So far so good.

But there are many natural operations on lists that $\star$GE can’t give us.

Given two finite lists π, ρ of proofs of *A*,
we can determine a particular finite list $\pi \mathbin{+\!\!+} \rho$
that is the concatenation of π then ρ.

Given what lists are, $\mathbin{+\!\!+}$ is perfectly sensible.
But we can't define $\mathbin{+\!\!+}$ with just $\star$GE.

The trouble is that $+\!\!+$ needs to recurse:

if π is empty, then $\pi +\!\!+ \rho$ is ρ,
and if π is nonempty, then $\pi +\!\!+ \rho$ is
$\star\mathrm{I}_{\mathrm{cons}}$ of π_h and $(\pi_t +\!\!+ \rho)$,
where π_h, π_t are π's head and tail.

$\star$GE lets us get to π_h and π_t,
but it doesn't let us use $+\!\!+$ in defining $+\!\!+$.
We'd need a proof that was a proper part of itself.

The problem is general.

Finite lists are defined like this in order to support recursion; most interesting operations on them are recursively defined.

(concatenation, reversing, adding a member to the end, filtering, summing, taking every other member, applying a function to each member,…)

But $\star$GE isn't up to the job.

Recursive operations on lists are justified by what lists are,
and what lists are is fully defined by $\star I_{nil}$, $\star I_{cons}$.

$\star$GE is locally sound, so justifiable.
And it's locally complete, but still not strong enough.

It gives us all the conclusions justifiable from the I rules,
but not all the justifiable proofs.

How to do better

$$\star\text{GE:}\quad \frac{A^\star \qquad C \qquad \begin{array}{c}\underbrace{[A],[A^\star]\quad}\\ \vdots \\ C\end{array}}{C}$$

$$\star\text{PE:}\quad \dfrac{A^\star \qquad C \qquad \begin{matrix}\underbrace{[A],[A^\star],(C)}\\ \vdots \\ C\end{matrix}}{C}$$

C is arbitrary, and might be A or $A^\star$;
in such cases, we need to keep track of how it's discharged:
qua C or qua $A/A^\star$.

The $\star\mathrm{I}_{\mathrm{nil}}$ case of local soundness is as for $\star$GE:

$$\begin{array}{ll} \star\mathrm{I}_{\mathrm{nil}}: \\ \star\mathrm{PE}: \end{array} \dfrac{\dfrac{}{A^\star} \quad \begin{array}{c}\vdots \\ C\end{array} \quad \begin{array}{c}\overbrace{[A],[A^\star],(C)}\\ \vdots \\ C\end{array}}{C} \quad \rightsquigarrow \quad \begin{array}{c}\vdots \\ C\end{array}$$

The $\star\mathrm{I}_{\mathrm{cons}}$ case is where the magic happens, and why the discharge distinction is needed:

$$\star\mathrm{I}_{\mathrm{cons}}:\ \star\mathrm{PE}:\ \dfrac{\dfrac{\begin{matrix}\vdots\\ A\end{matrix}\qquad\begin{matrix}\vdots\\ A^{\star}\end{matrix}}{A^{\star}}\qquad\begin{matrix}\vdots\\ C\end{matrix}\qquad\begin{matrix}\underbrace{[A],[A^{\star}],(C)}\\ \vdots\\ C\end{matrix}}{C}$$

$$\underbrace{\begin{matrix}\vdots\\ A\end{matrix}\qquad\begin{matrix}\vdots\\ A^{\star}\end{matrix}\qquad \star\mathrm{PE}:\ \dfrac{\begin{matrix}\vdots\\ A^{\star}\end{matrix}\qquad\begin{matrix}\vdots\\ C\end{matrix}\qquad\begin{matrix}\underbrace{[A],[A^{\star}],(C)}\\ \vdots\\ C\end{matrix}}{C}}\\ \vdots\\ C$$

Local completeness is basically as for ⋆GE, but with a new version as well:

$$\star\text{PE:}\quad \dfrac{A^\star \qquad \star\text{I}_{\text{nil}}:\ \dfrac{}{A^\star} \qquad \star\text{I}_{\text{cons}}:\ \dfrac{[A] \qquad [A^\star]}{A^\star}}{A^\star}$$

Local completeness is basically as for $\star$GE, but with a new version as well:

$$\star\text{PE:}\quad \frac{A^\star \qquad \begin{matrix}\star\text{I}_{\text{nil}}:\ \dfrac{}{A^\star}\end{matrix} \qquad \begin{matrix}\star\text{I}_{\text{cons}}:\ \dfrac{[A] \qquad (A^\star)}{A^\star}\end{matrix}}{A^\star}$$

$$\star\mathrm{I}_{\mathrm{nil}}: \quad \frac{}{A^\star} \qquad\qquad \star\mathrm{I}_{\mathrm{cons}}: \quad \frac{A \qquad A^\star}{A^\star}$$

This is a language for building proofs of $A^\star$.

$\star$PE let us translate this into a language for proofs of C if we can translate each rule.

The translation of $\star\mathrm{I}_{\mathrm{cons}}$ gets access to the tail both in the original and in translation.

$$\frac{}{C} \qquad\qquad \frac{A \qquad A^\star \qquad C}{C}$$

$$\star\mathrm{PE}:\ \dfrac{\star\mathrm{I}_{\mathrm{cons}}:\ \dfrac{\begin{matrix}\vdots\\ A\end{matrix}\qquad\begin{matrix}\vdots\\ A^\star\end{matrix}}{A^\star}\qquad\begin{matrix}\vdots\\ C\end{matrix}\qquad\begin{matrix}\underbrace{[A],[A^\star],(C)}\\ \vdots\\ C\end{matrix}}{C}$$

$$\begin{matrix}\underbrace{\begin{matrix}\vdots\\ A\end{matrix}\qquad\begin{matrix}\vdots\\ A^\star\end{matrix}\qquad \star\mathrm{PE}:\ \dfrac{\begin{matrix}\vdots\\ A^\star\end{matrix}\qquad\begin{matrix}\vdots\\ C\end{matrix}\qquad\begin{matrix}\underbrace{[A],[A^\star],(C)}\\ \vdots\\ C\end{matrix}}{C}}\\ \vdots\\ C\end{matrix}$$

$\star$PE gives us everything $\star$GE did immediately:
every application of $\star$GE is also an application of $\star$PE.

But unlike $\star$GE, $\star$PE lets us use the recursive structure of lists.

Because of this, we can use it to build proofs
that compute more functions on lists.

$$\star\text{CE}: \quad \dfrac{\begin{matrix}\pi \\ A^\star\end{matrix} \qquad \begin{matrix}\rho \\ A^\star\end{matrix} \qquad \star\text{I}_{\text{cons}}: \dfrac{[A] \qquad (A^\star)}{A^\star}}{A^\star}$$

This proof represents $\pi +\!\!\!+ \rho$.

(Exercise: what would it do with $[A^\star]$ instead?)

This proof puts the proof α of A at the end of the list π:

$$\mathbin{+\!\!+}: \quad \frac{\begin{array}{c}\pi \\ A^\star\end{array} \qquad \star\mathrm{I}_{\mathrm{cons}}: \dfrac{\begin{array}{c}\alpha \\ A\end{array} \qquad \star\mathrm{I}_{\mathrm{nil}}: \dfrac{}{A^\star}}{A^\star}}{A^\star}$$

Call it π **snoc** α

This proof reverses the list π:

$$\star\mathrm{PE}: \quad \frac{\begin{matrix}\pi \\ A^\star\end{matrix} \qquad \star\mathrm{I}_{\mathrm{nil}}: \dfrac{}{A^\star} \qquad \mathbf{snoc}: \dfrac{(A^\star) \qquad [A]}{A^\star}}{A^\star}$$

(Exercise: what would it do with $[A^\star]$ instead?)

We can take every other element of a list with ⋆PE.

If we can apply a function to an argument,
we can map that function over lists with ⋆PE.

If we can sum two numbers,
we can sum a list of numbers with ⋆PE.

And so on.

The situation

Given $\star I_{nil}$ and $\star I_{cons}$,
we have two candidate elimination rules: $\star$GE and $\star$PE.

Both are locally sound, so give only justifiable conclusions.
Both are locally complete, so give all justifiable conclusions.

But they're very different when we turn from provability to proof.
$\star$PE gives many more proofs, including many important ones.

$\star$PE is justified by the $\star$ I rules, I reckon.
But beyond that, what to make of this?

General elimination misses the target:
the rules it gives are sometimes too weak.

Local soundness and completeness are too coarse
for this kind of investigation.

Local completeness just tells us we have enough conclusions,
leaving open that we're missing many justifiable proofs.

(Local soundness seems to have the reverse problem:
it's compatible with local soundness that we have
unjustifiable proofs of justifiable conclusions.)

What can replace local soundness and completeness, though?

How to tell when a proof is justifiable on the basis of I rules?

Does $\star$PE get all of them?