A toolkit for metainferential logics

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Introduction

Some exciting recent work in higher metainferences:

•	BPS	'A hierarchy...'
		'(Meta)inferential levels...'
•	Pailos	'A fully classical...'
•	Scambler	'Classical logic...'
•	and more	

Much of this work is tied to particular languages, models, and logics.

But there is plenty of structure here,
already being put to good use in this work,
that is perfectly general.

My goal for this talk, then,
is to explore how much of this work can be done
as abstractly as possible.

In particular, I will reconstruct the ST hierarchy
and show that it matches two-valued classical logic
without mentioning values, connectives, etc until the very end

Throughout, the results are mostly not new;
they are lifted from the above-mentioned works.

The point is to see just how much structure
higher metainferences give us

For most of the talk, $\mathcal{L}$ is any language;
all I assume is that it is a set.

ℓ ranges over levels: $-1, 0, 1, 2, \ldots$

- A meta^{-1}inference is a member of $\mathcal{L}$
- A meta$^{\ell+1}$inference is $[\Gamma \succ \Delta]$,
  where Γ and Δ are sets of meta$^{\ell}$inferences

(Numbering in line with Pailos, not BPS/Scambler.)

These are the metainferences.

Counterexamples and consequence

I assume some fixed set of models.

A meta$^{\ell}$counterexample relation is:
a relation between models and meta$^{\ell}$inferences

A full counterexample relation is:
a relation between models and metainferences

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A full counterexample relation is:
a relation between models and metainferences

This is all ‘local’!

Given a full counterexample relation X and a level ℓ,
the meta$^{\ell}$counterexample relation $X(\ell)$ is:
the restriction of X in its codomain to meta$^{\ell}$inferences

We can give a full counterexample relation X
by specifying $X(\ell)$ for each level ℓ

Meta$^{\ell}$counterexample relations and full counterexample relations are all counterexample relations (XRs)

Given counterexample relation X, model $\mathfrak{m}$, and metainference μ, $\mathfrak{m}[\![\mathrm{X}]\!]\mu$ means that X relates $\mathfrak{m}$ to μ: the model is a counterexample to the metainference

(The brackets are to help keep our eyes from getting hairy.)

A meta$^\ell$consequence relation is a set of meta$^\ell$inferences

A full consequence relation is a set of metainferences.

Given a full consequence relation Σ and a level ℓ, the meta^{n}consequence relation $\Sigma(n)$ is Σ restricted to meta$^\ell$inferences

We can give a full consequence relation Σ by specifying $\Sigma(\ell)$ for each level ℓ.

Meta$^\ell$consequence relations and full consequence relations are all consequence relations (CRs)

Keeping an eye on both counterexample relations
and consequence relations is key.

Probably what we care about is consequence relations.

But much of the new metainferential technology
requires counterexample relations
due to the use of local validity

Given a meta$^\ell$counterexample relation X, the meta$^\ell$consequence relation $\mathcal{C}(X)$ is the set of meta$^\ell$inferences not in the image of X.

Given a full counterexample relation X, the full consequence relation $\mathcal{C}(X)$ is the set of metainferences not in the image of X.

It is familiar to fix a counterexample relation
and explore effects on consequence relations of restricting or expanding the class of models.

This is the reverse: our models are fixed,
and it is shifting counterexample relations that effects consequence.

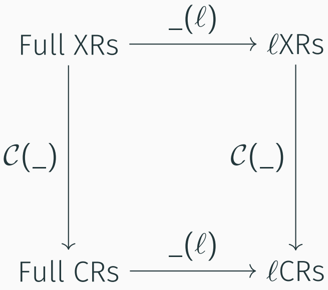
(Shifting counterexample relations
can simulate restricting models)

This all assumes nothing about the language, about models, etc.
(We don't even have monotonicity of consequence relations!)

But there's already enough here to see some structure
and prove some simple results.

Fact

For any full counterexample relation X and level ℓ,
$\mathcal{C}(X(\ell)) = \mathcal{C}(X)(\ell)$



Example

There can be distinct counterexample relations X, Y
such that $\mathcal{C}(X) = \mathcal{C}(Y)$.

(meta^{0}counterexample relations: ST and CL
full counterexample relations: ST_ω and $\widehat{CL}$)

If we care about counterexample:
giving just a consequence relation isn't enough.

If we care about consequence:
asking for a particular counterexample relation is asking too much

Connections between levels

Given a meta$^{\ell+1}$counterexample relation X,
its lowering $\downarrow$X is the meta$^{\ell}$counterexample relation such that
for any model $\mathfrak{m}$ and any meta$^{\ell}$inference μ,
$\mathfrak{m}[\![\downarrow \mathrm{X}]\!]\mu$ iff $\mathfrak{m}[\![\mathrm{X}]\!][\succ\mu]$.

Given a meta$^{\ell+1}$consequence relation Σ,
its lowering $\downarrow\Sigma$ is the meta$^{\ell}$consequence relation such that
for any meta$^{\ell}$inference μ,
$\mu \in \downarrow\Sigma$ iff $[\succ\mu] \in \Sigma$

Fact

For any meta$^{\ell+1}$counterexample relation X,
$\mathcal{C}(\downarrow X) = \downarrow \mathcal{C}(X)$.

$$\begin{array}{ccc}
(\ell+1)\text{XRs} & \xrightarrow{\;\downarrow_\;} & \ell\text{XRs} \\
\mathcal{C}(_)\big\downarrow & & \big\downarrow\mathcal{C}(_) \\
(\ell+1)\text{CRs} & \xrightarrow{\;\downarrow_\;} & \ell\text{CRs}
\end{array}$$

Given a meta$^{\ell}$counterexample relation X,
its lifting $\uparrow\mathsf{X}$ is the meta$^{\ell+1}$counterexample relation such that
for any model $\mathfrak{m}$ and any meta^{n+1}inference $[\Gamma \succ \Delta]$,
$\mathfrak{m}[\![\uparrow\mathsf{X}]\!][\Gamma \succ \Delta]$ iff:
there is no $\gamma \in \Gamma$ with $\mathfrak{m}[\![\mathsf{X}]\!]\gamma$, and $\mathfrak{m}[\![\mathsf{X}]\!]\delta$ for all $\delta \in \Delta$

Unlike lowering, we cannot lift consequence relations
in a way that matches lifing for counterexample relations.

There can be meta$^{\ell}$counterexample relations X and Y
with $\mathcal{C}(\mathrm{X}) = \mathcal{C}(\mathrm{Y})$ but $\mathcal{C}(\uparrow \mathrm{X}) \neq \mathcal{C}(\uparrow \mathrm{Y})$.
(At level 0, ST and CL are such.)

So there cannot be any operation $\uparrow$ on consequence relations
such that in general $\uparrow \mathcal{C}(\mathrm{X}) = \mathcal{C}(\uparrow \mathrm{X})$.

Lifting depends on information carried by a counterexample relation that is not there in the consequence relation it determines

Or: if someone specifies just a meta$^{\ell}$consequence relation, they have not thereby settled on any particular meta$^{\ell+1}$consequence relation

Lifting is a special case of slashing:

Given a meta$^{\ell}$counterexample relation X,
its lifting $\uparrow$X is the meta$^{\ell+1}$counterexample relation such that
for any model $\mathfrak{m}$ and any meta^{n+1}inference $[\Gamma \succ \Delta]$,
$\mathfrak{m}[\![\uparrow X]\!][\Gamma \succ \Delta]$ iff:
there is no $\gamma \in \Gamma$ with $\mathfrak{m}[\![X]\!]\gamma$, and $\mathfrak{m}[\![X]\!]\delta$ for all $\delta \in \Delta$.

Lifting is a special case of slashing:

Given two meta$^{\ell}$counterexample relations X and Y, their slashing X/Y is the meta$^{\ell+1}$counterexample relation such that for any model $\mathfrak{m}$ and any meta^{n+1}inference $[\Gamma \succ \Delta]$,
$\mathfrak{m}[\![X/Y]\!][\Gamma \succ \Delta]$ iff:
there is no $\gamma \in \Gamma$ with $\mathfrak{m}[\![X]\!]\gamma$, and $\mathfrak{m}[\![Y]\!]\delta$ for all $\delta \in \Delta$.

So $X^{\uparrow}$ is X/X

Slashing is key in work on higher metainferences.

Just as with lifting,
there is no corresponding operation on consequence relations.

This depends on the extra detail
carried by counterexample relations.

Fact

For any meta$^\ell$counterexample relations X, Y:
$\downarrow(X/Y) = Y$

Fact

So lowering is a retraction of lifting:
that is, for any meta$^\ell$counterexample relation X, we have $\downarrow(\uparrow X) = X$

Fact

Lifting is injective;
lowering is not injective and so not invertible

Excursion 1: more on slashing

Slashing has some exploitable structure

fact

$$(X/Z) \cup (Y/Z) \subseteq (X \cap Y)/Z$$
$$(X/Z) \cap (Y/Z) = (X \cup Y)/Z$$

fact

$$(Z/X) \cup (Z/Y) \subseteq Z/(X \cup Y)$$
$$(Z/X) \cap (Z/Y) = Z/(X \cap Y)$$

The following is enough to settle a great deal:

fact

If $X' \subseteq X$ and $Z \subseteq Z'$, then $X/Z \subseteq X'/Z'$

fact

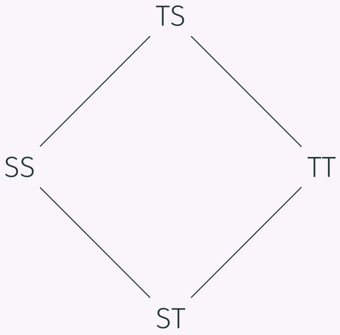
If $X/Z \subseteq X'/Z'$, then $X' \subseteq X$ and $Z \subseteq Z'$

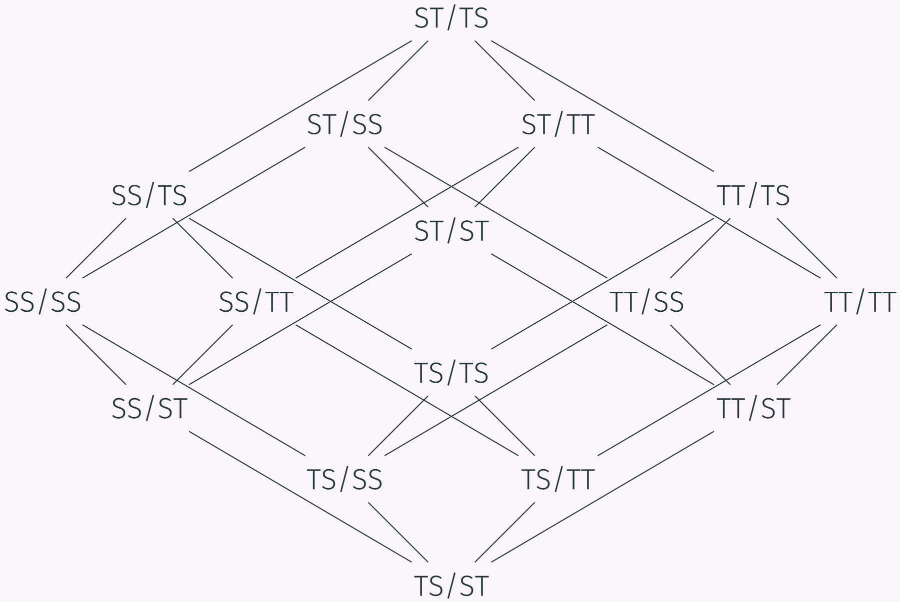
(So lifting is not monotonic)

For any XRs $T \subsetneq S$:

(More counterexamples at the top)

All are distinct;
all inclusions shown




ST/TS
ST/SS
ST/TT
SS/TS
TT/TS
ST/ST
SS/SS
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TT/SS
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SS/ST
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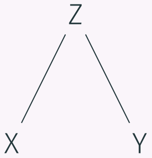
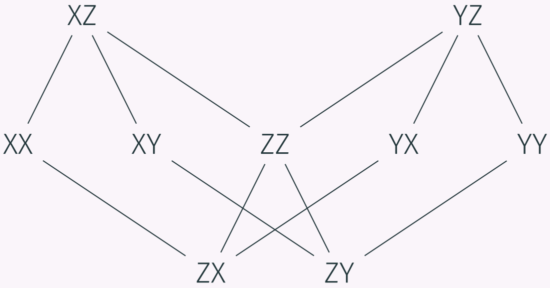
TS/SS
TS/TT
TS/ST

Or take X, Y, Z with



Up a level, we get this:



So to know the $\subseteq$ structure of all slashed XRs at any level,
it's enough to know the $\subseteq$ structure of a set they're built from

None of this depends at all on what language we're working with or what our models are

The definition of slashing all on its own does the work

Excursion 2: adjoints to lowering

When a monotonic function (like lowering)
has no inverse, there is sometimes a next-best:
perhaps it has an adjoint or two.

(Since these are posets,
adjunctions are monotone Galois connections.)

for any model $\mathfrak{m}$ and any meta$^{\ell}$inference μ,
$\mathfrak{m}[\![\downarrow X]\!]\mu$ iff $\mathfrak{m}[\![X]\!][\succ\mu]$.

It follows that $\downarrow$ is monotonic,
and that $\downarrow\bigcup X_i = \bigcup\downarrow X_i$

And it follows from that that $\downarrow$ is a left adjoint:
there is a $^{\uparrow o} : \ell XR \to (\ell+1)XR$
such that $\downarrow \dashv {}^{\uparrow o}$, which means $\downarrow X \subseteq Y$ iff $X \subseteq Y^{\uparrow o}$

$$Y^{\uparrow o} = \bigcup\{Z | \downarrow Z \subseteq Y\}$$

for any model $\mathfrak{m}$ and any meta$^{\ell}$inference μ,
$\mathfrak{m}[\![\downarrow X]\!]\mu$ iff $\mathfrak{m}[\![X]\!][\succ\mu]$.

It follows that $\downarrow$ is monotonic,
and that $\downarrow\bigcap X_i = \bigcap\downarrow X_i$

And it follows from that that $\downarrow$ is a right adjoint:
there is a $^{\uparrow i} : \ell XR \to (\ell + 1)XR$
such that $^{\uparrow i} \dashv \downarrow$, which means $Y^{\uparrow i} \subseteq X$ iff $Y \subseteq \downarrow X$

$$Y^{\uparrow i} = \bigcap\{Z | Y \subseteq \downarrow Z\}$$

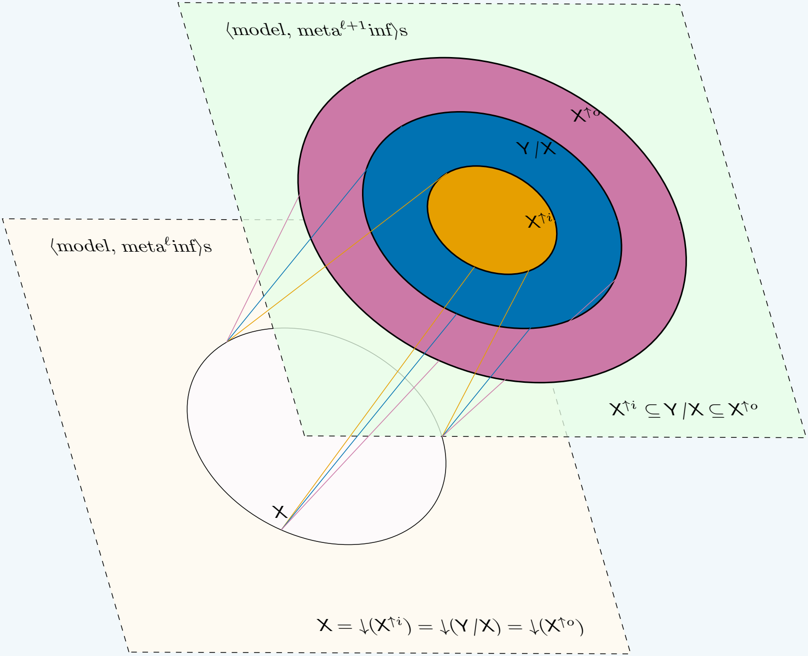
We have an adjoint situation ${}^{\uparrow i} \dashv \downarrow \dashv {}^{\uparrow o}$
(monotone Galois connection)

$\downarrow$ is not invertible, but ${}^{\uparrow i}$ and ${}^{\uparrow o}$ are kinda inversey to it.

$X^{\uparrow i}$ is the least Y with $X \subseteq \downarrow Y$,
and $X^{\uparrow o}$ is the greatest Y with $\downarrow Y \subseteq X$.

Since $\downarrow$ is surjective, we have $\downarrow(X^{\uparrow i}) = X = \downarrow(X^{\uparrow o})$,
and $X^{\uparrow i}$ and $X^{\uparrow o}$ are the least and greatest XRs that lower to X.

Recall that $\downarrow(Y/X) = X$ for any Y,
so $X^{\uparrow i} \subseteq Y/X \subseteq X^{\uparrow o}$


⟨model, meta$^{\ell+1}$inf⟩s
$X^{\uparrow o}$
Y / X
$X^{\uparrow i}$
⟨model, meta$^{\ell}$inf⟩s
X
$X^{\uparrow i} \subseteq Y / X \subseteq X^{\uparrow o}$
$X = \downarrow(X^{\uparrow i}) = \downarrow(Y / X) = \downarrow(X^{\uparrow o})$

If we think $\downarrow$ is onto something worth exploring,
and we want to think about natural ways of climbing up the levels,
$\uparrow^i$ and $\uparrow^o$ suggest themselves at least as much as $\uparrow$ does.

Example

Boolean bivaluations, with $\mathsf{CL}_{(-1)}$ the falsity relation.

$\mathcal{C}(\mathsf{CL}_{(-1)})$ is the set of classical theorems

$\mathcal{C}(\mathsf{CL}^{\uparrow}_{(-1)})$ is usual classical consequence

$\mathcal{C}(\mathsf{CL}^{\uparrow o}_{(-1)})$ validates $[\Gamma \succ \Delta]$ iff:
Γ is empty and $[\succ \Delta]$ classically valid

$\mathcal{C}(\mathsf{CL}^{\uparrow i}_{(-1)})$ validates $[\Gamma \succ \Delta]$ iff not:
Γ is empty and $[\succ \Delta]$ not classically valid

Excursions over!

Full counterexample relations

So far that’s all level by level,
or moving between adjacent levels.

But we can use it to get a look at full counterexample relations.

A full counterexample relation X is:

ℓ-downward coherent iff $\downarrow X(\ell') = X(\ell' - 1)$ for all $\ell' \leq \ell$

ℓ-upward coherent iff $\uparrow X(\ell') = X(\ell' + 1)$ for all $\ell' \geq \ell$

downward coherent iff ℓ-downward coherent for all ℓ

upward coherent iff ℓ-upward coherent for all ℓ

Given a ℓXR X, define the full XR $\widehat{X}$ by lifting and lowering.

Some authors identify X and $\widehat{X}$; I do not.
This is just one way to fit things together.

$$\widehat{X}(\ell) = \begin{array}{r} \vdots \\ \uparrow^n X \\ \vdots \\ \uparrow^2 X \\ \uparrow X \\ X \\ \downarrow X \\ \downarrow^2 X \\ \vdots \\ \downarrow^m X \\ \vdots \\ \downarrow^{\ell+1} X \end{array}$$

Given a ℓXR X, define the full XR $\widehat{\mathsf{X}}$ by lifting and lowering.

Some authors identify X and $\widehat{\mathsf{X}}$; I do not. This is just one way to fit things together.

$$\widehat{\mathsf{X}}(\ell+1) = \begin{array}{r} \vdots \\ \uparrow^n \mathsf{X} \\ \vdots \\ \uparrow^2 \mathsf{X} \\ \uparrow \mathsf{X} \\ \mathsf{X} \\ \downarrow \mathsf{X} \\ \downarrow^2 \mathsf{X} \\ \vdots \\ \downarrow^m \mathsf{X} \\ \vdots \\ \downarrow^{\ell+1} \mathsf{X} \end{array}$$

Given a ℓXR X, define the full XR $\widehat{X}$ by lifting and lowering.

Some authors identify X and $\widehat{X}$; I do not. This is just one way to fit things together.

$$\widehat{X}(\ell+2) \quad = \quad \begin{array}{r} \vdots \\ \uparrow^n X \\ \vdots \\ \uparrow^2 X \\ \uparrow X \\ X \\ \downarrow X \\ \downarrow^2 X \\ \vdots \\ \downarrow^m X \\ \vdots \\ \downarrow^{\ell+1} X \end{array}$$

Given a ℓXR X, define the full XR $\widehat{\mathsf{X}}$ by lifting and lowering.

Some authors identify X and $\widehat{\mathsf{X}}$; I do not. This is just one way to fit things together.

$$\begin{aligned} & & \vdots \\ \widehat{\mathsf{X}}(\ell + n) &= & {\uparrow^n} \mathsf{X} \\ & & \vdots \\ & & {\uparrow^2} \mathsf{X} \\ & & {\uparrow} \mathsf{X} \\ & & \mathsf{X} \\ & & {\downarrow} \mathsf{X} \\ & & {\downarrow^2} \mathsf{X} \\ & & \vdots \\ & & {\downarrow^m} \mathsf{X} \\ & & \vdots \\ & & {\downarrow^{\ell+1}} \mathsf{X} \end{aligned}$$

Given a ℓXR X, define the full XR $\widehat{X}$ by lifting and lowering.

Some authors identify X and $\widehat{X}$; I do not. This is just one way to fit things together.

$$\widehat{X}(\ell - 1) = \begin{array}{r} \vdots \\ \uparrow^n X \\ \vdots \\ \uparrow^2 X \\ \uparrow X \\ X \\ \downarrow X \\ \downarrow^2 X \\ \vdots \\ \downarrow^m X \\ \vdots \\ \downarrow^{\ell+1} X \end{array}$$

Given a ℓXR X, define the full XR $\widehat{X}$ by lifting and lowering.

Some authors identify X and $\widehat{X}$; I do not. This is just one way to fit things together.

$$\widehat{X}(\ell - 2) \quad = \quad \begin{array}{r} \vdots \\ \uparrow^n X \\ \vdots \\ \uparrow^2 X \\ \uparrow X \\ X \\ \downarrow X \\ \downarrow^2 X \\ \vdots \\ \downarrow^m X \\ \vdots \\ \downarrow^{\ell+1} X \end{array}$$

Given a ℓXR X, define the full XR $\widehat{X}$ by lifting and lowering.

Some authors identify X and $\widehat{X}$; I do not. This is just one way to fit things together.

$$\begin{array}{rcr} & & \vdots \\ & & \uparrow^n X \\ & & \vdots \\ & & \uparrow^2 X \\ & & \uparrow X \\ & & X \\ & & \downarrow X \\ & & \downarrow^2 X \\ & & \vdots \\ \widehat{X}(\ell - m) & = & \downarrow^m X \\ & & \vdots \\ & & \downarrow^{\ell+1} X \end{array}$$

Given a ℓXR X, define the full XR $\widehat{X}$ by lifting and lowering.

Some authors identify X and $\widehat{X}$; I do not. This is just one way to fit things together.

$$
\begin{array}{rcr}
 & & \vdots \\
 & & \uparrow^n X \\
 & & \vdots \\
 & & \uparrow^2 X \\
 & & \uparrow X \\
 & & X \\
 & & \downarrow X \\
 & & \downarrow^2 X \\
 & & \vdots \\
 & & \downarrow^m X \\
 & & \vdots \\
\widehat{X}(-1) & = & \downarrow^{\ell+1} X
\end{array}
$$

Fact

Where X is a meta$^{\ell}$counterexample relation, $\widehat{X}$ is downward coherent and ℓ-upward coherent

Fact

If a full counterexample relation Y is downward coherent and ℓ-upward coherent, then $Y = \widehat{Y(\ell)}$

Meta$^\ell$counterexample relations X, Y agree iff $\mathcal{C}(X) = \mathcal{C}(Y)$

Full counterexample relations X, Y
agree at level ℓ iff $X(\ell)$ and $Y(\ell)$ agree
They agree fully iff $\mathcal{C}(X) = \mathcal{C}(Y)$

Fact

If full counterexample relations X, Y
are ℓ-downward coherent and agree at level ℓ,
then they agree at all levels $m \leq \ell$

Example

The corresponding claim for ℓ-upward coherence is false

$\widehat{\mathrm{ST}}$ and $\widehat{\mathrm{CL}}$ are determined by meta^{0}counterexample relations, so are 0-upward coherent. They agree at level 0, but not at level 1

A full consequence relation Σ is self-obeying at level ℓ iff:
for every $[\Gamma \succ \phi] \in \Sigma(\ell + 1)$, if $\Gamma \subseteq \Sigma(\ell)$ then $\phi \in \Sigma(\ell)$.

A full consequence relation Σ is strongly self-obeying at level ℓ iff:
for every $[\Gamma \succ \Delta] \in \Sigma(\ell + 1)$, if $\Gamma \subseteq \Sigma(\ell)$ then $\Delta \cap \Sigma(\ell) \neq \emptyset$.

(Strong self-obedience is Scambler's 'closed [sic] under its own laws')

Self-obedience is more familiar than strong self-obedience.

Σ is self-obeying at level ℓ iff
$\Sigma(\ell)$ is closed under the operation
$C(\Pi) = \Pi \cup \{\phi | [\Gamma \succ \phi] \in \Sigma(\ell + 1) \text{ and } \Gamma \subseteq \Pi\}$

There is no closure operation
connected to strong self-obedience in this way

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Example

$\widehat{\mathrm{CL}}$ is self-obeying but not strongly self-obeying at level -1,
since $[p \vee \neg p \succ p, \neg p] \in \widehat{\mathrm{CL}}(0)$

fact

If a full counterexample relation X is ℓ-upward coherent, then $\mathcal{C}(X)$ is self-obeying at level n for all $n \geq \ell$

Example

Downward coherence does not suffice for self-obedience.
$\widehat{ST\lambda}$ is downward coherent, but not self-obeying at level -1

The abstract slash hierarchy

Consider any relation $\sqsubseteq$ on models

A counterexample relation X goes up $\sqsubseteq$ iff:
whenever $\mathfrak{m} \sqsubseteq \mathfrak{m}'$ and $\mathfrak{m}[\![X]\!]\mu$, then $\mathfrak{m}'[\![X]\!]\mu$

A counterexample relation X goes down $\sqsubseteq$ iff:
whenever $\mathfrak{m}' \sqsubseteq \mathfrak{m}$ and $\mathfrak{m}[\![X]\!]\mu$, then $\mathfrak{m}'[\![X]\!]\mu$

Fact

If X goes down $\sqsubseteq$ and Y goes up it,
then Y/X goes down it and X/Y goes up it.

Where X is a counterexample relation and $\mathfrak{M}$ a set of models, let $X|_{\mathfrak{M}}$ be the restriction of X to $\mathfrak{M}$.

A set $\mathfrak{M}$ of models is at the top of $\sqsubseteq$ iff for every model $\mathfrak{m}$ there is some $\mathfrak{m}' \in \mathfrak{M}$ with $\mathfrak{m} \sqsubseteq \mathfrak{m}'$

Fact

If $\mathfrak{M}$ is at the top of $\sqsubseteq$ and X goes up $\sqsubseteq$, then X agrees (fully) with $X|_{\mathfrak{M}}$

Suppose we have the following:

two meta^{-1}counterexample relations X and Y
and a set $\mathfrak{M}$ of models such that:

$$X|_{\mathfrak{M}} = Y|_{\mathfrak{M}}$$

$\mathfrak{M}$ is at the top of $\sqsubseteq$,
and X goes down $\sqsubseteq$ and Y goes up it.

This is enough for the key hierarchy result

Define:

- - $XY_{-1} = Y$
  - $YX_{-1} = X$
- - $XY_{\ell+1} = (YX_\ell)/(XY_\ell)$
  - $YX_{\ell+1} = (XY_\ell)/(YX_\ell)$

Let $XY_\omega(\ell) = XY_\ell$, and let $YX_\omega(\ell) = YX_\ell$

Fact

For every level ℓ, $\mathcal{C}(\mathsf{XY}_\ell) = \mathcal{C}(\widehat{\mathsf{X}|_{\mathfrak{M}}}(\ell)) = \mathcal{C}(\widehat{\mathsf{Y}|_{\mathfrak{M}}}(\ell))$

XY_ω agrees fully with $\widehat{\mathsf{X}|_{\mathfrak{M}}}$ $(= \widehat{\mathsf{Y}|_{\mathfrak{M}}})$

This gives a strategy for liberalizing a model theory
without affecting the resulting consequence relation at any level

Example hierarchies

Our language is a usual propositional language;
our total space of models is strong Kleene valuations;
$\mathfrak{M}$ is Boolean valuations;
$\sqsubseteq$ is information order;
S is having value $\neq 1$;
T is having value 0.

The following are immediate:
$\mathfrak{M}$ is at the top of $\sqsubseteq$;
S goes down $\sqsubseteq$ and T up it;
and $X|_{\mathfrak{M}} = Y|_{\mathfrak{M}}$

So ST_ω agrees fully with CL.

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So ST_ω agrees fully with CL.
The first-order extension is immediate.

The same for weak Kleene.

For any matrix consequence,
let all existing values be $\sqsubseteq$-incomparable,
and add a new value $*$ at the $\sqsubseteq$-bottom.

Extend existing operations to be $\sqsubseteq$-monotonic,
so $\sqsubseteq$ extends to models pointwise.

$\mathfrak{M}$ is the models that don't use $*$.
X is being undesignated in the old sense or having value $*$;
Y is being undesignated in the old sense.

Then XY_ω agrees fully with the original matrix consequence.

Other examples?

Conclusion

Much recent work on ST-style metainferential hierarchies
can be made fully general

The structure of metainferences themselves
is enough for abstract methods to get a grip

Slashing and lowering in particular seem interesting
in their own right, regardless of language or models

This allows for quick generalizations of known results

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Pailos, F. M. (2020). A fully classical truth theory characterized by substructural means. *The Review of Symbolic Logic*, 13(2):249–268.

Scambler, C. (2020). Classical logic and the strict tolerant hierarchy. *Journal of Philosophical Logic*, 49(2):351–370.