63 negations

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1024

1024 to 100

100 to 63

63

1024

DLL, SM

(Bounded) DLL:

Axioms:

$$A \vdash A$$

$$A \vdash \top \qquad \bot \vdash A$$

$$A_0 \wedge A_1 \vdash A_i \qquad A_i \vdash A_0 \vee A_1$$

$$A \wedge (B \vee C) \vdash (A \wedge B) \vee (A \wedge C)$$

Rules:

$$\frac{A \vdash B \qquad A \vdash C}{A \vdash B \wedge C} \qquad \frac{A \vdash C \qquad B \vdash C}{A \vee B \vdash C}$$

$$\frac{A \vdash B \qquad B \vdash C}{A \vdash C}$$

SM

$$\frac{A \vdash B}{-B \vdash -A}$$

Dualizing a sequent: swap premise/conclusion, $\wedge/\vee$, and $\top/\bot$.

Every axiom has a dual theorem. Every rule has a dual rule.

So every proof has a dual proof: a proof of the dual theorem.

Derived rule:

If $A \vdash B$ and $C(\ )$ is a positive context, then $C(A) \vdash C(B)$.
If $C(\ )$ is negative, then $C(B) \vdash C(A)$.

(Proof: induction on $C(\ )$.)

1024

10 principles

Normality principles:

N1: $\top \vdash -\bot$

N2: $-\top \vdash \bot$

N: N1 + N2

Antidistribution principles:

$$\text{A1: } -A \land -B \vdash -(A \lor B)$$
$$\text{A2: } -(A \land B) \vdash -A \lor -B$$
$$\text{A: A1 + A2}$$

Double negation principles:

$$\textsc{d}1: A \vdash --A$$

$$\textsc{d}2: --A \vdash A$$

$$\textsc{d}: \textsc{d}1 + \textsc{d}2$$

Minimal ex'ion principles:

x1: $A \wedge -A \vdash -\top$

x2: $-\bot \vdash A \vee -A$

x: x1 + x2

Recall: $-\top \vdash -B$, and $-B \vdash -\bot$.

Full ex'ion principles:

$x^{+}1$: $A \wedge -A \vdash \bot$

$x^{+}2$: $\top \vdash A \vee -A$

x^{+}: $x^{+}1 + x^{+}2$

The 1 principles and 2 principles are dual.

$2^{10} = 1024$ specifications.

How many distinct logics?

1024 to 100

1024 to 256

Clearly: X1 + N2 entails X^{+}1.
Clearly: X^{+}1 entails X1.
Less clearly: X^{+}1 entails N2.

$$\dfrac{\wedge R: \dfrac{-\top \vdash \top \qquad -\top \vdash -\top}{-\top \vdash \top \wedge -\top} \qquad x^{+}1: \dfrac{}{\top \wedge -\top \vdash \bot}}{-\top \vdash \bot}\ \text{Cut}$$

Dually, $\text{X}^{+}2$ entails N1.

No need for X^{+} principles.
Down to $2^8 = 256$.

1024 to 100

256 to 100

D*i* entails N*i*:

$$\dfrac{\dfrac{\bot \vdash -\top}{--\top \vdash -\bot} \qquad \top \vdash --\top}{\top \vdash -\bot}$$

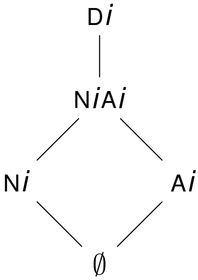
Di entails Ai:

Part I

$$\text{D2:}\quad \dfrac{\dfrac{-A \vdash -A \lor -B}{-(-A \lor -B) \vdash --A}}{-(-A \lor -B) \vdash A}$$

Part II

$$\text{D2:}\quad \dfrac{\dfrac{-(-A \lor -B) \vdash A \land B}{-(A \land B) \vdash --(-A \lor -B)}}{-(A \land B) \vdash -A \lor -B}$$


D*i*
N*i*A*i*
N*i*
A*i*
∅

This cuts $2^3 = 8$ down to 5.

256 was $(8 \times 2)^2$.

We reach $(5 \times 2)^2 = 100$.

100 to 63

Remaining entailments

That exhausts one-principle entailments.

Combinations remain.

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Combinations remain.

(Note: no use of distribution yet!)

Classical:

The pair of X^+ principles together entail all others.

So X + N does too. Usual presentation of Boolean algebra.

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The pair of X^+ principles together entail all others.

So $\text{X} + \text{N}$ does too. Usual presentation of Boolean algebra.

Of our 100, 15 are classical.

Full X brings full A.

Part I

x2: $\dfrac{}{-(A \wedge B) \vdash B \vee -B}$

Fiddling: $\dfrac{-(A \wedge B) \vdash B \vee -B}{-(A \wedge B) \vdash -(A \wedge B) \wedge (B \vee -B)}$

Dist: $\dfrac{-(A \wedge B) \vdash -(A \wedge B) \wedge (B \vee -B)}{-(A \wedge B) \vdash (-(A \wedge B) \wedge B) \vee (-(A \wedge B) \wedge -B)}$

DRule ($\wedge$elim): $\dfrac{-(A \wedge B) \vdash (-(A \wedge B) \wedge B) \vee (-(A \wedge B) \wedge -B)}{-(A \wedge B) \vdash (-(A \wedge B) \wedge B) \vee -B}$

Part II

x2: $\dfrac{}{-(A \wedge B) \vdash A \vee -A}$

Fiddling: $\dfrac{-(A \wedge B) \vdash A \vee -A}{-(A \wedge B) \wedge B \vdash (-(A \wedge B) \wedge B) \wedge (A \vee -A)}$

Dist: $\dfrac{-(A \wedge B) \wedge B \vdash (-(A \wedge B) \wedge B) \wedge (A \vee -A)}{-(A \wedge B) \wedge B \vdash (-(A \wedge B) \wedge B \wedge A) \vee (-(A \wedge B) \wedge B \wedge -A)}$

DR ($\wedge$E): $\dfrac{-(A \wedge B) \wedge B \vdash (-(A \wedge B) \wedge B \wedge A) \vee (-(A \wedge B) \wedge B \wedge -A)}{-(A \wedge B) \wedge B \vdash (-(A \wedge B) \wedge B \wedge A) \vee -A}$

DR (x1, fiddling): $\dfrac{-(A \wedge B) \wedge B \vdash (-(A \wedge B) \wedge B \wedge A) \vee -A}{-(A \wedge B) \wedge B \vdash -A \vee -A}$

Fiddling: $\dfrac{-(A \wedge B) \wedge B \vdash -A \vee -A}{-(A \wedge B) \wedge B \vdash -A}$

Full X plus Ni brings Di.

$$\dfrac{\dfrac{\dfrac{\dfrac{}{--A \wedge (A \vee -A) \vdash (--A \wedge A) \vee (--A \wedge -A)}\text{Dist}}{--A \wedge (A \vee -A) \vdash (--A \wedge A) \vee \bot}\text{D Rule } (X^{+}1 = X1N2)}{--A \wedge (A \vee -A) \vdash --A \wedge A}\bot\text{-drop}}{--A \wedge (A \vee -A) \vdash A}\wedge\text{-elim}$$

Full N plus Xi and A$\bar{i}$ brings Di.

$$
\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{}{A \wedge (-A \vee --A) \vdash (A \wedge -A) \vee (A \wedge --A)}\ \text{Dist}}{A \wedge (-A \vee --A) \vdash --A}\ \text{Fiddling (DRule, X1, N2)}}{A \wedge -(A \wedge -A) \vdash --A}\ \text{DRule (A2)}}{A \wedge -\bot \vdash --A}\ \text{DRule (X1,N2)}}{A \wedge \top \vdash --A}\ \text{DRule (N1)}}{A \vdash --A}\ \top\text{-drop}
$$

Di plus X$\bar{i}$ brings Xi.

$$\dfrac{\text{DR(N1):}\quad \dfrac{\text{SM:}\quad \dfrac{\text{X2:}\quad \dfrac{}{-\bot \vdash A \vee -A}}{-(A \vee -A) \vdash --\bot}}{-(A \vee -A) \vdash -\top}}{}$$

$$\text{Cut:}\ \dfrac{\text{Fiddling:}\ \dfrac{A \vdash --A}{-A \wedge A \vdash -A \wedge --A} \qquad \text{A1:}\ \dfrac{}{-A \wedge --A \vdash -(A \vee -A)}}{-A \wedge A \vdash -(A \vee -A)}$$

Xi plus N$\bar{i}$ plus A$\bar{i}$ brings Ai.

Part I

$$
\dfrac{\dfrac{\dfrac{\dfrac{}{-A \wedge --A \rhd -(A \vee B)}\ \text{X1}}{(-A \wedge --A) \vee (-A \wedge -(A \vee B)) \rhd -(A \vee B)}\ \text{Fiddling}}{-A \wedge (--A \vee -(A \vee B)) \rhd -(A \vee B)}\ \text{Dist}}{-A \wedge -(-A \wedge (A \vee B)) \rhd -(A \vee B)}\ \text{DRule (A2)}
$$

Part II:

$$
\dfrac{\dfrac{\dfrac{}{A \wedge -A \rhd B}\ \text{ECQ (X1 + N2)}}{(-A \wedge A) \vee (-A \wedge B) \rhd B}\ \text{Fiddling}}{-A \wedge (A \vee B) \rhd B}\ \text{Cut (Dist)}
$$

That's it!

Down to 63.

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Some interesting critters

CL is NX (= DX).

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FDE is D.

Preminimal logic (the logic of compatibility frames) is N1A1.

Dual premin (the logic of exhaustiveness frames) is N2A2.

Ockham logic (the logic of the Routley star) is NA.

D1X1 is the strongest logic here sound for minimal logic, N2D1X1 for intuitionist.

D2X2 is the strongest logic here sound for dual minimal logic, N1D2X2 for dual intuitionist.

D1X1 is the strongest logic here sound for minimal logic,
N2D1X1 for intuitionist.

D2X2 is the strongest logic here sound for dual minimal logic,
N1D2X2 for dual intuitionist.

Are they these logics?

Other than CL, there are two “attractive” logics among the 100:

L1:

X2D1 = XN1 = XA2D1

L2:

X1D2 = XN2 = XA1D2

Anybody recognize these?

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Organized by x

No X: 25 logics

~~X~~	0	N1	A1	N1A1	D1
0	⋆	⋆	⋆	⋆(Pre)	⋆(Quas)
N2	⋆	⋆	⋆	⋆	⋆
A2	⋆	⋆	⋆	⋆	⋆
N2A2	⋆(DPre)	⋆	⋆	⋆(Ock)	⋆
D2	⋆(DQuas)	⋆	⋆	⋆	⋆(FDE)

X1: 17 logics

X1	0	N1	A1	N1A1	D1
0	⋆	⋆	⋆	⋆	⋆ (Min)
N2	⋆	⋆	⋆	⋆	⋆ (Int)
A2	⋆	⋆	⋆	⋆	⋆
N2A2	A1	D1	⋆	D1	⋆
D2	X2A1 (L2)	X2A1 (C)	X2 (L2)	X2 (C)	X2 (C)

X2: 17 logics

X2	0	N1	A1	N1A1	D1
0	⋆	⋆	⋆	A2	X1A2 (L1)
N2	⋆	⋆	⋆	D2	X1A2 (C)
A2	⋆	⋆	⋆	⋆	X1 (L1)
N2A2	⋆	⋆	⋆	D2	X1 (C)
D2	⋆ (DMin)	⋆ (DInt)	⋆	⋆	X1 (C)

X: 4 logics

X	0	N1	A1	N1A1	D1
0	A	D1 (L1)	A2	D1 (L1)	A2 (L1)
N2	D2 (L2)	D (C)	D2 (L2)	D (C)	D2 (C)
A2	A1	D1 (L1)	$\star$	D1 (L1)	$\star$ (L1)
N2A2	D2 (L2)	D (C)	D2 (L2)	D (C)	D2 (C)
D2	A1 (L2)	D1 (C)	$\star$ (L2)	D1 (C)	$\star$ (C)

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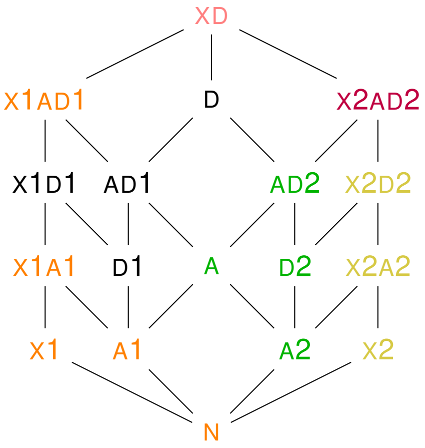
Organized by N


XD
X1AD1
D
X2AD2
X1D1
AD1
AD2
X2D2
X1A1
D1
A
D2
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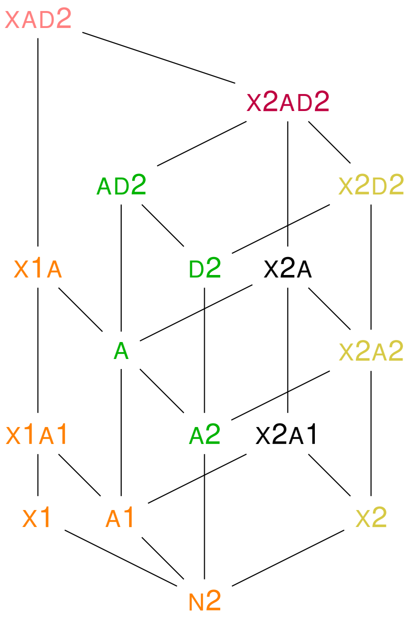
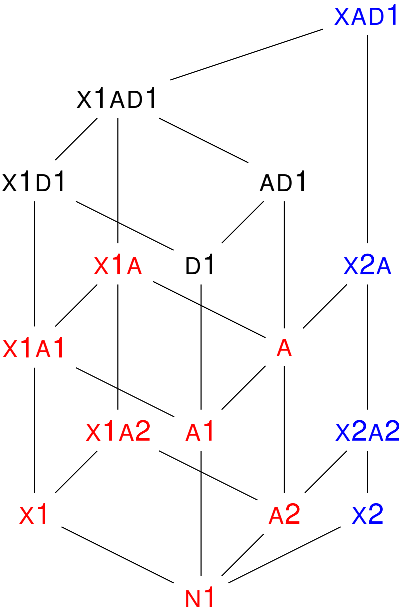
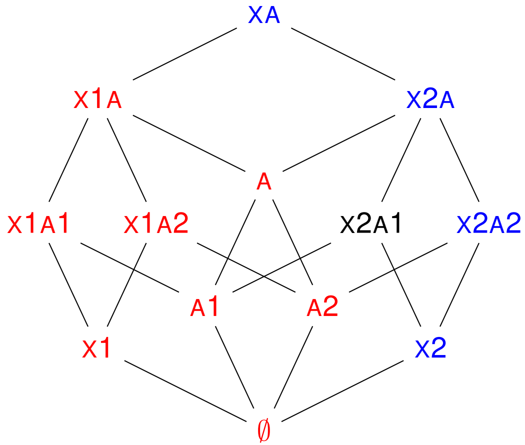
A1
A2
X2
N


XAD2
X2AD2
AD2
X2D2
X1A
D2
X2A
A
X2A2
X1A1
A2
X2A1
X1
A1
X2
N2


XAD1
X1AD1
X1D1
AD1
X1A
D1
X2A
X1A1
A
X1A2
A1
X2A2
X1
A2
X2
N1


XA
X1A
X2A
A
X1A1
X1A2
X2A1
X2A2
A1
A2
X1
X2
∅

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Next steps

Consider the rules of antilogism:

Antilogisms

$$\frac{A \wedge B \vdash C}{A \wedge \neg C \vdash \neg B} \qquad \frac{A \vdash B \vee C}{\neg B \vdash \neg A \vee C}$$

$$\frac{A \wedge B \vdash C \vee D}{A \wedge \neg C \vdash \neg B \vee D}$$

Now how many?

Which of these critters can coexist?

Which of these critters can coexist?

How do they relate to other critters?